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Are Instantons Found?

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It is argued that the recent prediction of the a dependence of the coupling, $g(a)$, of an effective lattice gauge theory for quarkless quantum chromodynamics is confirmed by a recent numerical evaluation of $g(a)$ for a latticized SU(2) gauge theory, thus providing strong evidence for the dynamical role of instantons. Additional predictions, testing the validity of the quantum chromodynamics bag model, are presented.

In a series of papers[1] we have argued that in quarkless quantum chromodynamics (QCD), instantons, because they induce a large coupling-constant renormalization and because their density increases rapidly with size, produce an abrupt transition at a well-defined distance scale from weak- to strong-coupling behavior. Recently,[2] with the aid of a comparison lattice gauge theory, we were able to make this notion precise enough to make a rough calculation of the static tension and radius of the flux tube joining a heavy quark pair. Unfortunately, these predictions cannot be directly tested since the real world contains three light quarks whose presence, we believe, has a sizable effect on the physics of QCD. In lieu of *accelerator* experiments we must turn to computer *experiments* (i.e., numerical studies of a lattice version of QCD) to test our ideas. Creutz[3] has recently performed such calculations for a pure SU(2) gauge theory. His "experimental" results are in such striking agreement with our "theoretical" predictions[2] that we feel that *they provide strong evidence for the fundamental dynamical role of instantons in the physics of gauge theories*. (We should point out that Kogut, Pearson, and Shigemitsu[4] have done a totally different sort of QCD "experiment"—a Padé extrapolation of the strong-coupling expansion—whose results are also quite consistent[2] with our picture.) In this Letter we shall compare theory[2] and experiment,[3] make some additional predictions, and discuss the implications of all this for the real world [SU(3) and three light quarks].

In Ref. 2 we studied the behavior of $g(a)^{-2}$, the coefficient of the Wilson term [$S_W = \sum_p \mathrm{Tr}(\Pi_p U_{\mathrm{link}})$, p for plaquette] in the effective lattice action which results from integrating out all degrees of freedom except those associated with the links of a lattice of spacing a. Although there are many other terms, we focus on S_W since it governs the continuum limit (small a, weak coupling) as well as the behavior of large planar Wilson loops (large a, strong coupling). For sufficiently small a, $g(a)$ must equal $g_{\rm af}(a)$, the perturbative asymptotic-freedom coupling constant. As a increases, $g(a)$ must be renormalized above $g_{\rm af}(a)$ by nonperturbative fluctuations whose importance increases rapidly with scale size. We believe, for reasons laid out in Ref. 2, that for a significant range of lattice spacings the net effect is a multiplicative renormalization of $g_{\rm af}$: $g^2(a)=\mu(a)g_{\rm af}{}^2(a)$, where $\mu(a)$ is the permeability of an instanton gas whose scale-size integration is cut off at $\rho_c\sim a$. The instanton gas is only meant to represent the nature of fluctuations in the theory on scales *less* than a. It has a chance of being accurate so long as a is small enough that the gas is reasonably dilute. If, within this limit, one finds a significant range of a where $g^2(a)$ exhibits the a dependence of the strong-coupling limit, then the string tension can be determined from the strong-coupling relation, $\sigma=\ln g^2(a)/a^2$.

In an SU(2) gauge theory, the permeability satisfies

$$\mu=\eta+(\eta^2+1)^{1/2},$$
$$\eta(a)=\tfrac{4}{3}\pi^2\int_0^{\rho_c}(d\rho/\rho)D(\rho)x(\rho),$$
$$x(\rho)=\tfrac{22}{3}\ln(1/\Lambda\rho), \qquad (1)$$

where $D(\rho)$ is the density of instantons of scale size ρ. For an isolated instanton, $D_0(\rho)C_Lx^4(\rho)e^{-x(\rho)}$, where $C_L=1.66\times10^4$ for the lattice definition[2] of g^2. In an instanton gas, however, the effect of interactions with all other instantons is to *increase* the single-instanton density above $D_0(\rho)$, and we need some quantitative representation of this effect. In Ref. 1 we showed that in the simple approximation in which the instanton sits in a cavity in a continuous medium of high permeability (Onsager's approximation for a dipolar medium), the e^{-x} in D_0 is replaced by $e^{-\alpha x}$, where α depends on cavity radius and instanton scale size. A physically reasonable choice of these parameters yields $\alpha=\frac{3}{4}$ and we therefore take $D(\rho)=C_Lx^4(\rho)e^{-3x(\rho)/4}$.

We can now evaluate $g(a)$ and check whether the string tension, $\sigma=\ln g^2(a)/a^2$, approaches a constant for large a.[5] As in the case of SU(3),[2] we find that once a is greater than a critical value $\bar{a}$, σ is essentially constant indicating that the instantons have produced, for $a\geq\bar{a}$, a strong-coupling lattice theory. The value of σ depends on $\lambda=\rho_c/a$ (to a very good approximation physical lengths, e.g., $\bar{a}$, scale like λ^{-1}, and physical energies, e.g., σ, scale like λ), and ranges from $(171\Lambda)^2$ for $\lambda=\frac{2}{3}$ to $(249\Lambda)^2$ for $\lambda=1$. We argued in Ref. 2 that this is the physically sensible range for ρ_c. To make concrete comparisons with Creutz,[3] we will take ρ_c in the middle of this range, i.e., $\lambda=\frac{5}{6}$. In Fig. 1 we show σa^2 as a function of $\beta=4/g^2(a)$ for $\lambda=\frac{5}{6}$. This plot (the same display used by Creutz) indicates an almost instantaneous transition from weak coupling, $\sigma a^2=\exp[-(6\pi^2/11)(\beta-1.97)]$, to strong coupling, $\sigma a^2=\ln g^2(a)=-\ln\beta/4$, at $\bar{a}=0.0039\Lambda^{-1}$. ($\bar{a}$ is the point at which the extrapolated weak- and strong-coupling curves intersect.) The "dilute gas" validity criterion for our model is definitely satisfied for $\beta\gtrsim1$, since the integrated fraction of space-time occupied by instantons is 0.015 at $\beta=1$. For smaller β and higher density, the dipole-gas approximation to the instanton medium begins to fail.

Let us now summarize Creutz's results.[3] Using Monte Carlo techniques, he has evaluated Wilson loop expectation values in an SU(2) lattice gauge theory[6] on a 10^4-site periodic lattice. The Wilson loop expectation is found for different loop sizes and values of $\beta=4/g^2$. Creutz calculates the string tension, for a fixed value of

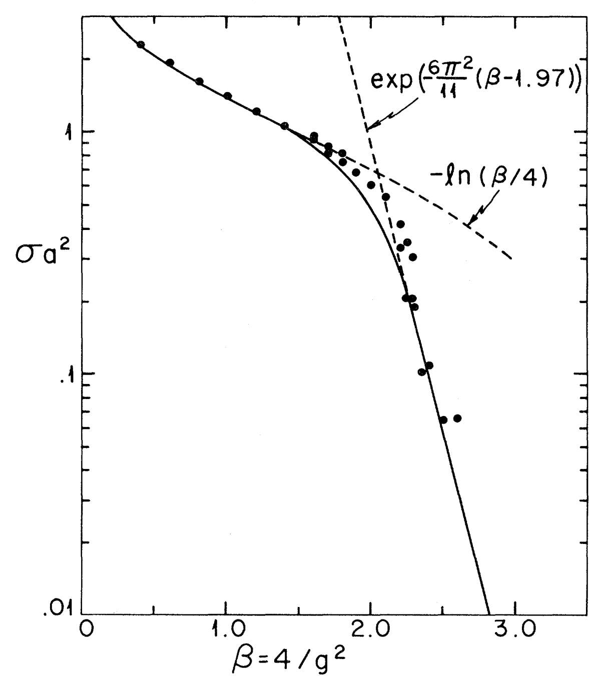

FIG. 1. Coupling strength vs lattice spacing squared (in units of σ^{-1}). The solid line is our solution, the dots are Creutz's data, and the dashed lines are the expected strong- and weak-coupling limits.

β, by fitting the loop expectations to an area law (which is quite visible for all but small coupling). This determines σa^2 as a function of β. With the assumption that σ is a constant, this determines g as a function of a. For strong coupling, i.e., small β, one necessarily finds $\sigma a^2 = \ln g^2(a)$. The acid test of the physical relevance of the lattice gauge theory, beyond its ability to produce a constant string tension,[7] is the requirement that $g(a)$ join smoothly onto asymptotic-freedom behavior for weak coupling. Thus for large β one should find $[\sigma a^2]_{\rm af} = \exp[-(6\pi^2/11)\times(\beta-\beta_0)]$, where β_0 determines the ratio of $\sqrt{\sigma}$ to the continuum renormalization scale parameter Λ: $(6\pi^2/11)\beta_0 = \ln(\sigma\Lambda^2)$.

Creutz's results (the points in Fig. 1) clearly show a sharp transition at $\beta = 2$ from strong-coupling to asymptotic-freedom behavior. His quoted (but uncertain to within a factor of 2) value for $\sqrt{\sigma}$ is 200Λ. Creutz's transition seems even sharper than ours, but since our couplings need agree only in the strong- and weak-coupling limits, this may not be significant. The agreement with our predictions is encouraging and seems to confirm the major dynamical role of instantons in quarkless QCD. There are, of course, some uncertainties in the "theoretical predictions" (and to a lesser degree in the "experimental results"). The curve of σa^2 vs β hardly changes as λ varies from $\frac{2}{3}$ to 1 (although $\sqrt{\sigma}$ scales as λ). The results are much more sensitive to our representation of the instanton interactions: If $D(\rho)$ is replaced by $D_0(\rho)$ $(e^{-3x/4}\to e^{-x})$ the value of β_0 changes from 2 to 1.5 and $\sqrt{\sigma}$ decreases by a factor of 4!

What is invariant to these uncertainties is the existence of an abrupt transition between weak- and strong-coupling behavior in the neighborhood of $\beta = 2$ and a string tension which is a very large multiple of Λ^2. Roughly speaking, the relation between σ and the value of g at which the transition occurs (g_0) is $\sigma = \Lambda^2 \exp(24\pi^2/11 g_0^2)$: Tiny errors in determining g_0 are amplified into big errors in σ. For this reason one can think of the Monte Carlo calculation as providing a means of fine-tuning the qualitative, but physically transparent, instanton method.

We have argued previously[1] that our semiclassical treatment of QCD produces a physical flux tube between heavy quarks. This notion was not used in the above discussion, but it can be tested with our (or Creutz's) evaluation of $g(a)$. In Ref. 2 we argued that the flux-tube radius, R, must be less (greater) than the value of a at which one just enters (leaves) the strong- (weak-) coupling regime. Since the transition is very abrupt, the uncertainty in R is small ($\pm 20\%$), and for definiteness we can choose $R = \bar{a} = 0.0039\Lambda^{-1}$. On the other hand, if the QCD bag model is correct, the string tension satisfies $\sigma = \pi R^2 E_c^2$, where E_c is the effective field strength within the tube. By flux conservation, E_c satisfies [for SU(2)] $E_c \times \pi R^2 = (3/4)^{1/2} g(R)$, where $g(R)$ is the static quark coupling strength. The coupling appropriate to a quark source in a flux cylinder of radius R was computed in Ref. 2 for SU(3). The corresponding SU(2) result is

$$8\pi^2/g^2(R) = \tfrac{22}{3}\ln[4.72/R(19.2\Lambda)].$$

Taken together these equations imply that $\sigma^{1/2}R = (3/4\pi)^{1/2} g(R)$. This relation can be tested by setting $R = \bar{a}$, and using either our or Creutz's results. We find that $\sigma^{1/2}\bar{a}/(3/4\pi)^{1/2} g(\bar{a}) = 1.02$ and that to within 5%, this result is independent of λ or of our treatment of instanton interactions. Given the intrinsic uncertainty in the determination of R this is excellent agreement, and strong evidence for the accuracy of the ACD bag picture.

One can obtain direct evidence for the QCD bag model by using computer experiments to explore the spatial structure of the flux tube directly. To do this one should evaluate the correlation of a large planar Wilson loop, W_L (thought of as producing a flux tube) with a small, single-plaquette, Wilson loop $W(\mu\nu)$ (thought of as a dipole probe that measures a lattice approximation to $\mathrm{Tr}F_{\mu\nu}^2$) [$(\mu\nu)$ indicate the plaquette orientation]. To explore the spatial structure of the flux tube one should choose the large loop as big as possible, and vary the orientation and location of the small loop with respect to the large loop. In order to get an adequate approximation to the continuum field distribution the lattice spacing should be small compared with $\bar{a}$. The expectation value of the small loop in the presence of the large loop [taken to lie in the (03) plane], $\langle W_L W(\mu\nu)\rangle/\langle W_L\rangle$, is proportional to $\mathrm{Tr}\,F_{\mu\nu}^2$. At a given point one can measure the components of the electric field, E_i, and the magnetic field, $\epsilon_{ijk}B_k$, by using loops in the $(0i)$ or (ij) planes, respectively.

In the semiclassical QCD bag model one expects to find a nonvanishing imaginary electric field, equal to iE_c, inside the flux tube. A particularly advantageous way of testing this is to measure $F = [\mathrm{Tr}(\vec{B}^2 - \vec{E}^2)]^{1/4}$ as a function of transverse distance from the large loop. First, F

vanishes in the vacuum state and no subtraction is required in the continuum limit. Second, many fluctuations, such as uncorrelated instantons, do not contribute to $\vec{B}^2 - \vec{E}^2$. We expect it to equal $E_c^{1/2}$ for $d=0$, and to drop sharply to zero for $d>R$.

Therefore, we expect that (1) for plaquettes in the plane of the large loop, F^2 should equal E_c, which, according to the bag-model argument, equals $(4/3)^{1/2}\sigma$; (2) the radius of the flux tube, R, defined as the ratio of the area under the curve of F versus transverse distance to the central value of F, should equal $R_{\rm bag} = (3/4\pi)^{1/2} g(R)\sigma^{-1/2} = 0.77\sigma^{-1/2}$. These predictions should be easy to check on the computer, are insensitive to both the uncertainty in the value of λ and the treatment of instanton interactions, and thus should provide strong tests of the QCD bag model. In addition, the measurement of individual components of the field strength, $\mathrm{Tr}F_{\mu\nu}^2$, should yield valuable information on the structure of the flux-tube surface.

Finally, we would like to comment on the sensitivity of our picture of QCD to changing the gauge group and to adding light-quark flavors. We expect the physics to be fairly insensitive to the gauge group: The instanton method, as described in this paper, yields, when applied to SU(3) rather than SU(2), virtually the same value of σ (210Λ) and a slightly smaller value of the coupling at which the transition between week and strong coupling occurs ($g^2 \sim 1.5$ rather than 2). There should soon be a Monte Carlo calculation to compare with this prediction. On the other hand we do expect light quarks to have at least one important quantitative effect: In the absence of chiral symmetry breaking, light quarks strongly suppress instantons. But instantons themselves provide a mechanism for chiral-symmetry breaking[1,8] and generate a dynamical quark mass which increases rapidly with increasing distance scale and eventually "turns on" the instanton component of vacuum fluctuations. The net effect is to *increase* the distance scale, in units of Λ^{-1}, of the weak- to strong-coupling transition and to *decrease* the dimensionless string tension, $\sigma^{1/2}\Lambda$. A rather large effect in this direction is needed in order to agree with what is known experimentally about σ and Λ. The string tension has long been known to be about (430 MeV)^2, while observed scaling violations suggest a Λ of 10 to 20 MeV (remember that we are using a lattice Λ which is approximately 20 times smaller than the more conventional perturbation-theory Λ). Therefore $\sigma^{1/2}/\Lambda$ in the real world is somewhere between 20 and 40, whereas in quarkless QCD both "theory" and "experiment"[3] yield a value of about 200. It is therefore a major challenge to any proponent of QCD to explain this fact, i.e., that *light quarks must produce a large increase in the length scale of the transition from weak to strong coupling.*

At the moment we have only very rough qualitative information, but it seems quite possible that including light quarks in our semiclassical treatment will preserve the feature of a very sharp transition between weak and strong coupling while substantially increasing the length scale of that transition. We are trying to make this argument quantitative in order eventually to compare our predictions with real, as opposed to computer, experiments! This research was supported in part by the National Science Foundation under Grant No. PHY78-01221 and in part by the U. S. Department of Energy under Grant No. EY-76-02-2220.

[1]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 17, 2717 (1978), and 19, 1826 (1979).

[2]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 20, 3279 (1979).

[3]M. Creutz, to be published.

[4]J. Kogut, R. Pearson, and J. Shigemitsu, Phys. Rev. Lett. 43, 484 (1979).

[5]For SU(N) the string tension is given by $\ln[Ng^2(a)]/a^2$, for all $N>2$. Our instanton method works and yields a string tension $\sigma \approx (200\Lambda)^2$ for all $2 \leq N \leq 10$.

[6]K. Wilson, Phys. Rev. D 10, 2445 (1975).

[7]It is conceivable, although unlikely, that QCD confines color, yet the string tension varies like a^{-p} ($0 \leq p < 1$), for large a, corresponding to a confining potential that increases as R^{1-p}. Creutz's results indicate that $p=0$.

[8]D. Caldi, Phys. Rev. Lett. 39, 121 (1977); R. Carlitz, Phys. Rev. D 17, 3225 (1978).