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Dynamic Scaling of Ultrasonic Attenuation at the Liquid Helium λ Point

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We present a new approach to the ultrasonic attenuation in liquid helium, based on the Pippard-Buckingham-Fairbanks relations. An appropriate frequency-dependent generalization yields a theory with no adjustable parameters, in good agreement with experiment.

In spite of many detailed and extensive experimental studies[1-5] of the ultrasonic attenuation at the λ point of liquid helium there does not yet exist a satisfactory theory of this interesting critical phenomenon. The purpose of this note is to provide such a theory. The numerous theoretical contributions to this subject,[6-9] although failing to provide a definitive theory, have suggested some useful qualitative ideas which have been helpful to the experimentalists in attempting to fit their data phenomenologically. With the increasing accuracy and range of the measurements this approach has become, however, less than satisfactory. The exponent $1+y$ describing the dependence of the attenuation on frequency at the λ point has been found[1] to be significantly greater than 1. Furthermore, different values of y are needed for fitting the data in the low-[2] and high-frequency[1] ranges. We cite two more failures: (1) For temperatures T different from the λ-point value T_λ the data do not scale accurately[1,2] according to the reduced frequency variable $\Omega=\omega/\Gamma$, where ω is the angular frequency of the ultrasound and Γ is a T-dependent relaxation rate. (2) The "symmetrical" assumption for subtracting the fluctuation portion of the attenuation at $T<T_\lambda$ yields unphysical negative values[1] for the residual Landau-Khalatnikov[10] portion. There is clearly a need for a reliable theory which can serve as a guide for further exploration of the attenuation of ultrasound in liquid helium.

We start with the thermodynamic expression for the isentropic compressibility of liquid helium in the vicinity of the λ point,

$$\beta_S=\beta_T-\frac{T\alpha_T^{\,2}}{C_P}=\text{const}+\frac{\text{const}}{C_P}, \tag{1}$$

where α_T, β_T, and C_P are the thermal expansion coefficient, the isothermal compressibility, and the specific heat at constant pressure, respectively. The final form of Eq. (1) is an approximation that holds in the immediate vicinity of the λ point. It follows from the Pippard-Buckingham-Fairbank relations and results from the fact that α_T, β_T, and C_P all have the same asymptotic critical behavior. These leading terms cancel from Eq. (1), leaving the *net critical variation entirely contained in the denominator* of the small final term.

Because of the smallness of this term, we can linearize with respect to it in the thermodynamic expression for $v = u_\lambda + u_1$, the $\omega = 0$ sound velocity of sound, obtaining the fractional change away from the λ point as

$$u_1/u_\lambda = C_0/\tilde{C}_P, \tag{2}$$

where[11,12] $C_0 = 4.0\times10^{-2}$. The normalized critical specific heat is for $T > T_\lambda$

$$\tilde{C}_P = \ln(t_0/t), \tag{3}$$

where $t = (T - T_\lambda)/T_\lambda$ and[13] $t_0 = 0.25$. Because of the very small value of the specific heat exponent the logarithmic representation of Eq. (3) will be sufficiently accurate for our purposes. We now note that the thermodynamic dependence of Eq. (3) on temperature can be reexpressed in terms of a t-dependent rate, $\Gamma(t)$, provided this functional dependence is specified. In a temperature regime where a power law holds we have

$$\Gamma = \Gamma_0(t/t_0)^\varphi, \tag{4}$$

where φ and Γ_0 are constants. Γ_0 sets the frequency scale. Elimination of t by substitution of Eq. (4) into Eq. (3) puts Eq. (2) into the form

$$\frac{u_1}{u_\lambda} = \frac{C_1}{\ln(\Gamma_0/\Gamma)} \equiv \frac{C_1}{L_1}, \tag{5}$$

where $C_1 = \varphi C_0$ and $L_1 \equiv \ln(\Gamma_0/\Gamma)$. The thermodynamic critical dependence is now expressed entirely in terms of the relaxation rate Γ. The basic idea of dynamic scaling[6] is that when ω, the frequency of the disturbance to which the system is submitted, exceeds Γ, then the latter is no longer relevant and should be replaced by ω. With this approach we see that the variables of Eq. (5) become the frequency-dependent quantities $u_1(\Gamma,\omega)$ and $L_1(\Gamma,\omega)$. The departure from the $\omega = 0$ thermodynamic limit and the passage to the λ point is effected, according to the dynamic scaling idea, by

$$L_1(0,\omega) \approx L_1(|\omega|, 0) = \ln(\Gamma_0/|\omega|). \tag{6}$$

Substituting Eq. (6) into the ω-dependent generalization of Eq. (5) and neglecting for the moment the imaginary part, we find the λ-point dispersion

$$\frac{u_1(0,\omega)}{u_\lambda} = \frac{C_1}{L_1(0,\omega)} \approx \frac{C_1}{\ln(\Gamma_0/|\omega|)}. \tag{7}$$

According to Eq. (7) the higher the frequency the more the velocity is raised above its λ-point value, in complete accord with the experimental findings.[2,5,11] But because the attenuation data are both more precise and more extensive, we turn immediately to u_2, the imaginary part of the complex velocity $u = u_1 + iu_2$. The generalization of Eq. (5) to a complex velocity is

$$u(\Gamma,\omega)/u_\lambda = C_1/L(\Gamma,\omega), \tag{8}$$

where $L = L_1 + iL_2$ is proportional to the frequency-dependent specific heat. Being a causal linear response function, $L(\Gamma,\omega)$ is an analytic function of ω in the upper half of the complex frequency plane. In the $\Gamma = 0$ limit discussed above, inspection (or application of the Kramers-Kronig relations) yields

$$L(0,\omega) \approx \ln\frac{\Gamma_0}{-i\omega} = \ln\frac{\Gamma_0}{\omega} + i\frac{\pi}{2} \tag{9}$$

for $\omega > 0$, so that the required imaginary part is $L_2 = \pi/2$. The attenuation at the λ point is therefore given by

$$-\frac{u_2(0,\omega)}{u_\lambda} = -C_1 \,\mathrm{Im}\frac{1}{L} = \frac{C_1 L_2}{L_1^2 + L_2^2} \approx \frac{(\pi/2)C_1}{[\ln(\Gamma_0/\omega)]^2 + \pi^2/4}. \tag{10}$$

The frequency dependence of Eq. (10) is contained entirely in the denominator. The effective frequency-dependent exponent determined from the slope of a log-log plot is consequently the logarithmic derivative

$$y(\omega) = \frac{d\ln|L|^{-2}}{d\ln\omega} = \frac{2L_1}{|L|^2} \approx \frac{2}{\ln(\Gamma_0/|\omega|)}. \tag{11}$$

At relatively low frequencies $\omega/2\pi < 1$ MHz the logarithm is typically ≈ 12, giving $y \approx \frac{1}{6} = 0.17$. In the very-high-frequency range the logarithm drops by a factor of 2, raising the effective exponent to $y \approx \frac{1}{3} = 0.33$, in agreement with the findings of Tozaki and Ikushima.[1]

A quantitative comparison of Eq. (10) with experiment requires information on C_1 and Γ_0. We obtain this from our previous studies[14-17] of the thermal conductivity. We found that $\Gamma/2\pi > 1$ MHz is a van Hove–type precritical range, where $\Gamma = 2B_\psi\kappa^2$. The background Onsager coefficient at saturated vapor pressure is $B_\psi = 1.2\times10^{-4}$ cm^2 sec^{-1}. The inverse correlation length is $\kappa = \kappa_0 t^{2/3}$, where the scale factor is $\kappa_0 = 0.7\times10^8$ cm^{-1}. It follows that $\Gamma = (1.2\times10^{12}\ \mathrm{sec}^{-1})t^{4/3}$. Comparison with Eq. (4) gives $\Gamma_0/2\pi = 30.0$ GHz and $\varphi = \frac{4}{3}$, so

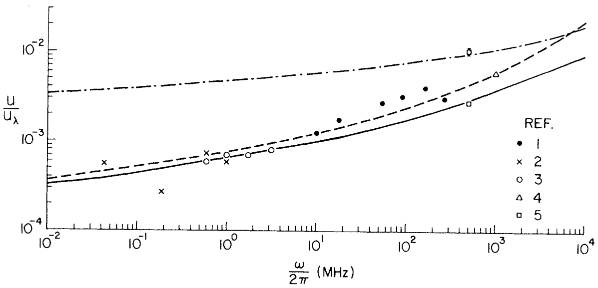

FIG. 1. Fractional change u/u_λ in sound velocity at the λ point vs frequency $\omega/2\pi$ in megahertz. The real and negative imaginary parts, u_1/u_λ and $-u_2/u_\lambda$ shown by the dot-dashed and solid curves, correspond to dispersion and attenuation, respectively. The dashed curve shows the "minimal" theory of attenuation.

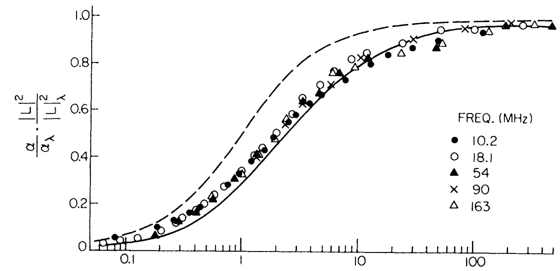

FIG. 2. "Processed" attenuation vs scaled frequency $\Omega=\omega/\Gamma$ for five runs at the frequencies $(\omega/2\pi)$ shown. The curve is the predicted scaling function $F(\Omega)$ of Eq. (17).

that $C_1=\frac{4}{3}C_0=5.3\times10^{-2}$. The numerical coefficient in the numerator of Eq. (10) is $(\pi/2)C_1=0.083$. Using this coefficient and the above value of Γ_0 we have plotted Eq. (10) as the dashed line in Fig. 1. Clearly in evidence is the monotonically increasing slope discussed above in connection with Eq. (11). The data (as identified in the legend) not only exhibit this trend, but also agree satisfactorily in absolute magnitude with Eq. (10). It should be emphasized that our theory is of the "no-fit" type. *The agreement has resulted without any adjustable parameter.* All of the information required for normalizing the theory has come from other sources.

In order to go beyond the above "minimal" theory and predict the ultrasonic attenuation in the entire Γ-ω plane we need to take into account how the various modes of wave number k and relaxation rate $\Gamma(k,\kappa)=2B_\psi(k^2+\kappa^2)$ are contributing to the frequency-dependent specific heat. The ϵ expansion provides a systematic framework for such a calculation. We report here on the $\epsilon=0$, or $D=4$ (four-dimensional), limit of such a calculation. Although only the first step in a complete theory, the $D=4$ limit already gives a reasonable result. Using a sharp upper cutoff, Γ_c, and including a background term we find

$$L(\Gamma,\omega)=\int_\Gamma^{\Gamma_c}\frac{d\Gamma'}{\Gamma'-i\omega}\left(1-\frac{\Gamma}{\Gamma'}\right)+B$$
$$=\ln\frac{\Gamma_c}{\Gamma-i\omega}-i\frac{\Gamma}{\omega}\ln\left(1-\frac{i\omega}{\Gamma}\right)+B, \qquad (12)$$

where we assume both Γ and ω to be small compared to Γ_c. The specific-heat data[13,18] indicate for the background $B=1.45$. Requiring the $\omega\to0$ thermodynamic limit of Eq. (12) to agree with $\ln(\Gamma_0/\Gamma)$ yields $\Gamma_c=e^{1-B}\Gamma_0=19.1$ GHz. The real and imaginary parts of Eq. (12) are

$$L_1(\Gamma,\omega)=\ln\frac{\Gamma_c}{(\Gamma^2+\omega^2)^{1/2}}-\frac{1}{\Omega}\tan^{-1}\Omega+B \qquad (13a)$$

and

$$L_2(\Gamma,\omega)=\tan^{-1}\Omega-\frac{1}{2\Omega}\ln(1+\Omega^2), \qquad (13b)$$

where the latter is a function only of the scaled frequency $\Omega=\omega/\Gamma$. At the λ point $L_2(0,\omega)=\pi/2$, as in Eq. (9). Equation (6) and the real part of Eq. (9) possess, however, only logarithmic accuracy, and are replaced by the $\Gamma=0$ limit of Eq. (13a). The resulting prediction for the attenuation is shown in Fig. 1 by the solid curve, somewhat below the predictions of the "minimal" theory (dashed curve). The corresponding dispersion is shown by the dot-dashed curve. The predicted attenuation and dispersion are both in excellent agreement with the light-scattering data[5] at 0.5 GHz. The attenuation reported by Commins and Rudnick[4] at 1 GHz is, however, somewhat larger than predicted.

Using the general form of Eq. (10), $C_1L_2/|L|^2$, and substituting Eqs. (13a) and (13b) gives the attenuation for arbitrary Γ and ω. Equation (13a) depends upon both Ω and the "distance" in the Γ-ω plane, $(\Gamma^2+\omega^2)^{1/2}$. To exhibit the scaling properties of Eq. (13b) it is therefore necessary to eliminate $|L|^2$ from the denominator of Eq. (10) and to multiply the data[1] α/α_λ (normalized to unity at the λ point) by $|L|^2/|L|_\lambda^2$. The resulting "processed" data are plotted in Fig. 2 versus Ω. The curve shows the scaling function $(2/\pi)L_2$ predicted from Eq. (13b). The scatter of the data for $\Omega\gg1$ may be attributable to the difficulty in reaching thermal equilibrium. (A kind of a hysteresis was reported in this range.) For Ω of the order of unity and for "processed" data in the range $0.2 \leq (\alpha/\alpha_\lambda)|L/L_\lambda|^2 \leq 0.5$ the data points coalesce and depend only upon Ω, within an accuracy of roughly 5%. In view of the variation of ω from 10.2 to 163 MHz, this coalescence is a confirmation of scaling. In this, the inclusion of the "correction factor" $|L/L_\lambda|^2$ is vital. Without it the scatter would be roughly 30%. Such a scatter is evident in Fig. 3 of Ref. 1 where an attempt was made to scale α/α_λ without any correction factor such as we have derived here.

We further remark in Fig. 2 that the trend of the scaled data is in good agreement with the shape of the predicted curve. A renormalization of the temperature scale by about 20% would shift the curve enough to bring it into exact accord with the data in the $0.2 < \Omega < 0.5$ range. We also note that in the linear range $\Omega < 1$ the second term in Eq. (13b) cancels one-half of the first term. The variation of L_2 with Ω is therefore significantly slower than that of the first term alone as shown by the dashed curve in Fig. 2. The agreement with the data seems to confirm this slower variation and in turn to confirm the spectral function in the integrand of Eq. (13).

The treatment for $T < T_\lambda$ follows similar lines. For the sake of brevity we report here only the salient features, with a more complete account to be published elsewhere. Equation (8) continues to apply, but L has in addition to Eq. (12) a Landau-Khalatnikov[9] portion proportional to $\Gamma_1/(\Gamma_1 - i\omega)$, where Γ_1 is the order-parameter relaxation rate. The $|L|^2$ denominator in Eq. (10) mixes the two contributions so that the attenuation cannot be split into two distinct portions. Only after multiplication of the data by $|L|^2$ does this become possible, with a scaling behavior revealed for each portion separately.

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