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Quark-Glue Structure of the η and η' Mesons, with Application to $\psi \to \eta(\eta')\gamma$ and $\psi' \to \psi\eta$

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A careful analysis of the Ward identities involving the axial divergences $\partial^\mu A_\mu$, combined with a fit to the 2γ decays of the π^0, η, and η' via the quantum electrodynamic anomaly, allows a consistent description of the pseudoscalar nonet with zero (or small) quantum chromodynamic surface term. The gluonic matrix elements $\langle 0|F\tilde{F}|P\rangle$ of the η and η' obtained from the Ward identities lead to a good quantitative understanding of the relative rates $(\psi\to\eta'\gamma)/(\psi\to\eta\gamma)$, and a qualitative understanding of the nonsuppression of $\psi'\to\psi\eta$.

The U(1) problem[1] is of great interest in modern particle theory because it provides a vital link between quantum chromodynamics (QCD) and the phenomenological successes of chiral SU(2) $\otimes$ SU(2) [or SU(3) $\otimes$ SU(3)] current algebra. Briefly, the problem may be stated as follows: Why do the low-lying pseudoscalars not display the same quark-model structure as the vector nonet; a nonstrange quartet $(u\bar{d}, d\bar{u}, (u\bar{u}\pm d\bar{d})/\sqrt{2})$ comprising the π^+, π^0, and a fourth neutral pseudoscalar (p^0) roughly degenerate with the π (Ref. 2); a heavier, strange quartet $(u\bar{s}, d\bar{s}, s\bar{d}, s\bar{u})$; and an even heavier neutral containing hidden strangeness ($\bar{s}s$). Instead, the lightest neutral except for the π^0 is the $\eta(549)$ which is even heavier than the K's, and is therefore not a candidate for the p^0. Moreover, the decay rate for $\eta\to 2\gamma$ as calculated through the Adler-Bell-Jackiw anomaly[3] seems to support an octet *quark* structure for the η. Certainly the $\eta'(958)$ is not a candidate for p^0.

The QCD version of the Adler-Bell-Jackiw anomaly has been proposed by 't Hooft[4] as a starting point for the resolution of the U(1) problem. Exactly how this is implemented is not clear, although several recent works on the U(1) problem have presented some arguments utilizing the concept of quark-gluon mixing in the mass matrix[5] or estimating the QCD surface term[1] by use of the $1/N$ expansion.[6]

In this work[7] I shall return to a careful analysis of the relevant Ward identities and of the 2γ data, and show that the observed masses and the 2γ decay rates for the η and η' can be supported *without the introduction of a substantial QCD surface term.* An important result of this analysis will be that the coupling of the η (as well as the η') to two gluons is substantial. This has some striking phenomenological implications for the decays $\psi\to\eta'(\eta)\gamma$ and $\psi'\to\psi\eta$, which will be discussed. Finally, as another result of the analysis, it will be seen that the short-distance structure of the η' is characterized by a large probability density of light (u and d) quarks and gluons, about ten times higher than that in the η or π^0.

Prerequisite for all that follows is a careful definition of all states and couplings. I shall define the usual nine axial-vector quark currents on a basis of three quark flavors $[q=(u,d,s)]$

$$A_\mu{}^a \equiv \bar{q}\gamma_\mu\gamma_5(\lambda_a/2)q,\quad a=0\ldots 8, \tag{1}$$

where $\lambda_0=(\frac{2}{3})^{1/2}\,\underline{1}$, and $\underline{1}$ is the unit matrix.

I define as well the partially conserved U(1) "symmetry current"[1]

$$\tilde{A}_\mu{}^0 = A_\mu{}^0 - (\tfrac{3}{2})^{1/2} K_\mu, \tag{2}$$

where

$$\partial^\mu K_\mu = (g^2/16\pi^2) F_{\rho\sigma}{}^a \tilde{F}_a{}^{\rho\sigma}. \tag{3}$$

The divergence of $\tilde{A}_\mu{}^0$ is soft, i.e.,

$$\tilde{D}^0 \equiv \partial^\mu \tilde{A}_\mu{}^0 = i\bar{q}\gamma_5\{\tfrac{1}{2}\lambda^0, \mathfrak{M}\} q. \tag{4}$$

Here $\mathfrak{M}$ is the quark mass matrix. The other eight divergences are given by formulas similar to (4),

$$D^a \equiv \partial^\mu A_\mu{}^a = i\bar{q}\gamma_5\{\tfrac{1}{2}\lambda^a, \mathfrak{M}\} q, \quad a = 1\ldots 8. \tag{5}$$

The Ward identities are derived by considering the propagators at zero momentum transfer,

$$\Delta^{ab}(0) \equiv i\int d^4x \langle T D^a(x) D^b(0)\rangle_{\text{vac}}, \qquad a,b = 1\ldots 8, \tag{6}$$

and similar definitions for $\Delta^{a0}(0)$ [with $D^b(0) \to \tilde{D}^0(0)$] and Δ^{00} [with $D^a(x)D^b(\tilde{0}) \to \tilde{D}^0(x)\tilde{D}^0(0)$].

Integration by parts, and the use of Eqs. (4), (5), and the canonical commutators leads to the result

$$\Delta^{ab}(0) = -\tfrac{1}{4}\langle \bar{q}\{\lambda^a, \{\lambda^b, \mathfrak{M}\}\} q\rangle_{\text{vac}} + \delta_{a0}\delta_{b0}\mathcal{S}, \qquad a,b = 0\ldots 8, \tag{7}$$

where $\mathcal{S} \equiv -i(\tfrac{3}{2})^{1/2}\int d^4x\, \partial^\mu \langle T K_\mu(x) \tilde{D}^0(0)\rangle_{\text{vac}}$ is a possible surface term arising from the nontrivial vacuum topology of QCD.[1]

At this point, we assume (for simplicity) that $m_u = m_d$ and define the nonzero couplings to the *physical* nonet [i.e., *there are no mixing angles or mass matrices and no assumptions about SU(3) invariance*]:

$$\langle \pi^b | \partial^\mu A_\mu{}^a | 0\rangle = F_\pi m_\pi{}^2 \delta_{ab}, \quad a,b = 1,2,3, \tag{8}$$

$$\langle K^b | \partial^\mu A_\mu{}^a | 0\rangle = F_K m_K{}^2 \delta_{ab}, \quad a,b = 4,5,6,7, \tag{9}$$

$$\langle \eta | \partial^\mu A_\mu{}^8 | 0\rangle = F_{8\eta} m_\eta{}^2, \tag{10}$$

$$\langle \eta | \partial^\mu A_\mu{}^0 | 0\rangle = F_{0\eta} m_\eta{}^2, \tag{11}$$

$$\langle \eta | \partial^\mu \tilde{A}_\mu{}^0 | 0\rangle = \tilde{F}_{0\eta} m_\eta{}^2, \tag{12}$$

and analogous equations for $\langle \eta' | \partial^\mu A_\mu{}^8 | 0\rangle$, $\langle \eta' | \partial^\mu A_\mu{}^0 | 0\rangle$, and $\langle \eta' | \partial^\mu \tilde{A}_\mu{}^0 | 0\rangle$.

If the quantities $\Delta^{ab}(0)$ in Eq. (7) are now saturated with the nonet, one obtains[8] after some rearrangement

$$(F_{8\eta} m_\eta)^2 + (F_{8\eta'} m_{\eta'})^2 = \tfrac{4}{3}(F_K m_K)^2 - \tfrac{1}{3}(F_\pi m_\pi)^2 + \tfrac{4}{3}\delta, \tag{13}$$

$$F_{8\eta}\tilde{F}_\eta m_\eta{}^2 + F_{8\eta'}\tilde{F}_{\eta'} m_{\eta'}{}^2 = (F_\pi m_\pi)^2, \tag{14}$$

$$(\tilde{F}_\eta m_\eta)^2 + (F_{\eta'} m_{\eta'})^2 = 3(F_\pi m_\pi)^2 + 2\mathcal{S}, \tag{15}$$

where $F_\eta \equiv (\sqrt{2}\tilde{F}_{0\eta} + F_{8\eta})$, $\tilde{F}_{\eta'} = \sqrt{2}\tilde{F}_{0\eta'} + F_{8\eta'}$, and $\delta = \tfrac{1}{2}(m_s - m_u)(\langle \bar{u}u\rangle_0 - \langle \bar{s}s\rangle_0)$.

To connect with previous work[2] note that a *possible* solution of Eqs. (13)–(15) (with $\mathcal{S} = 0$) is of the (undesirable) vector nonet type: $m_\eta = m_\pi$, $m_{\eta'}{}^2 = 2m_K{}^2 - m_\pi{}^2$, and $\sqrt{2}\tilde{F}_{0\eta'} = \sqrt{2}F_{8\eta} = \tilde{F}_{0\eta} = -F_{8\eta'} = (\tfrac{2}{3})^{1/2}F_\pi = (\tfrac{2}{3})^{1/2}F_K$, $\delta = 0$. Without the anomaly (i.e., $\tilde{F}_{0\eta} = F_{0\eta}$, etc.) this corresponds to $\eta \sim (\bar{u}u + dd)/\sqrt{2}$, $\eta' \sim \bar{s}s$.

It will be the aim of the subsequent discussion (i) to examine Eqs. (13)–(15) for alternate solutions which are consistent with the known mass spectrum and with any other relevant constraints on the F's (such as the 2γ decays); (ii) to obtain information on the gluonic couplings $(g^2/16\pi^2) \times \langle 0 | F_{\mu\nu}{}^a \tilde{F}_a{}^{\mu\nu} | \eta, \eta'\rangle$ which will allow some useful statements to be made with respect to ψ decays into η and η'; and (iii) to obtain some insight into the dynamical structure of the η and η' mesons.

In order to explore the alternate solutions, more input is required, and I first turn to an analysis of the radiative decays $\pi^0, \eta, \eta' \to 2\gamma$. Consider the equations expressing the presence of the *electromagnetic* anomaly[3] for the currents $A_\mu{}^3$, $A_\mu{}^8$, and $A_\mu{}^0$ (no tilde). From Eqs. (2), (4), (5) one has

$$\partial^\mu A_\mu{}^a = D^a + S_a \mathfrak{A}_{\text{em}}, \quad a = 3,8,0, \tag{16}$$

where $D^a = i\bar{q}\gamma_5\{\lambda^a/2, \mathfrak{M}\} q + \delta^{a0}(\tfrac{3}{2})^{1/2}\partial^\mu K_\mu$, $\mathfrak{A}_{\text{em}} \equiv (\alpha_{\text{em}}/4\pi) F_{\mu\nu}{}^{\text{em}} \tilde{F}_{\text{em}}{}^{\mu\nu}$, and $(S_3, S_8, S_0) = (\sqrt{3}/18)(\sqrt{3}, 1, 2\sqrt{2})$.

Since $A_\mu{}^0$ (not $\tilde{A}_\mu{}^0$) is multiplicatively renormalized,[1] one expects (D^3, D^8, D^0) to provide proper interpolators[3] for a linear combination of the renormalized π^0, η, η' fields. I express this as

$$\begin{pmatrix} D^3 \\ D^8 \\ D^0 \end{pmatrix} = \begin{pmatrix} F_\pi m_\pi{}^2 & 0 & 0 \\ 0 & F_{8\eta} m_\eta{}^2 & F_{8'} m_{\eta'}{}^2 \\ 0 & F_{0\eta} m_\eta{}^2 & F_{0\eta'} m_{\eta'}{}^2 \end{pmatrix} \begin{pmatrix} \varphi_{\pi^0}{}^{\text{ren}} \\ \varphi_\eta{}^{\text{ren}} \\ \varphi_{\eta'}{}^{\text{ren}} \end{pmatrix}, \tag{17}$$

where it is important to note that $(F_{0\eta}, F_{0\eta'}) \neq (\tilde{F}_{0\eta}, \tilde{F}_{0\eta'})$. The quantities without tildes $(F_{0\eta}, F_{0\eta'})$ were defined in Eq. (11) *et seq.*

Proceeding as in Ref. 3, one finds as a result of the low-energy theorems for $\langle 2\gamma|(\Box+m_\eta^2)\varphi_\eta^{\rm ren}|0\rangle$, etc., that

$$A_{\eta'\to 2\gamma}:A_{\eta\to 2\gamma}:A_{\pi^0\to 2\gamma}=(F_{8\eta}S_0-F_{0\eta}S_8):(F_{0\eta'}S_8-F_{8\eta'}S_0):(F_{8\eta}F_{0\eta'}-F_{0\eta}F_{8\eta'})S_3/F_\pi, \quad (18)$$

where $A_{p\to 2\gamma}$ is defined by $\langle 2\gamma|H|p\rangle=A_{p\to 2\gamma}\epsilon_{\mu\nu\rho\sigma}\times\epsilon_1^{\mu}\epsilon_2^{\nu}k_1^{\rho}k_2^{\sigma}$. Experimentally,[9] the relationship is

$$A_{\eta'\to 2\gamma}:A_{\eta\to 2\gamma}:A_{\pi^0\to 2\gamma}=1.9:1.0:1.3. \quad (19)$$

Equation (18) is in excellent agreement with these data for the assignments

$$F_{8\eta}=F_{0\eta'}=1.1F_\pi,$$
$$F_{8\eta'}=-F_{0\eta}=-0.17F_\pi. \quad (20)$$

Equation (20) is the basis for the usual statement that the η is primarily octet in its quark content.

If we insert Eq. (20) into the basic set (13)–(15), and use the known particle masses, we find,[10] *for* $\mathcal{S}=0$,

$$\delta=-[F_K^2-(1.1F_\pi)^2]m_K^2 \text{ [from Eq. (13)]}, \quad (21)$$
$$\bar{F}_{\eta'}\equiv\sqrt{2}\tilde{F}_{0\eta'}+F_{8\eta'}=\pm 0.24F_\pi, \quad (22)$$
$$\bar{F}_{\eta}\equiv\sqrt{2}\tilde{F}_{0\eta}+F_{8\eta}=(0.06\pm 0.11)F_\pi, \quad (23)$$

where the ± signs in Eqs. (22) and (23) are correlated. It will be seen below, in our consideration of $\psi\to\eta\gamma,\eta'\gamma$ decays that the upper sign in Eqs. (22) and (23) is preferred.

I now use these values to obtain the gluonic couplings. Taking matrix elements of the divergence of Eq. (2) and using Eqs. (20), (22), and (23), one obtains[11]

$$A_\eta=(\tfrac{2}{3})^{1/2}(F_{0\eta}-\tilde{F}_{0\eta})m_\eta^2=0.67F_\pi m_\eta^2,$$
$$A_{\eta'}=(\tfrac{2}{3})^{1/2}(F_{0\eta'}-\tilde{F}_{0\eta'})m_{\eta'}^2=0.66F_\pi m_{\eta'}^2, \quad (24)$$

where $A_{\eta(\eta')}=(g^2/16\pi^2)\langle 0|F_\mu^{a}\tilde{F}_a^{\mu\nu}|\eta(\eta')\rangle$. I now turn to some applications of these results.

(1) One may postulate that the amplitudes for the Okubo-Zweig-Iizuka nonconserving decays $\psi\to\eta\gamma(\eta'\gamma)$ are proportional to the gluonic matrix elements $A_\eta(A_{\eta'})$. [I have in mind a process $\psi\to 2g\gamma\to(\eta \text{ or } \eta')\gamma$.] In that case

$$\frac{\Gamma(\psi\to\eta'\gamma)}{\Gamma(\psi\to\eta\gamma)}=\left(\frac{A_{\eta'}}{A_\eta}\right)^2\left(\frac{k_{\eta'\gamma}}{k_{\eta\gamma}}\right)^3=7.3 \quad (25)$$

in good agreement with the recently measured[12] ratio 5.75 ± 1.42.[13]

(2) The sizable gluonic matrix element A_η can also be used to provide some insight into a long-standing puzzle, namely, the large size of the SU(3)-nonconserving decay $\psi'\to\psi\eta$. Phase space and angular momentum considerations alone would seem to be able to account for the factor of 10 in the ratio $\Gamma(\psi'\to\psi\eta)/\Gamma(\psi'\to\psi\pi\pi)$.[14] Standard mixing methods would then give an extra factor of $(\sin\theta)^2=0.01$, where θ is the singlet-octet mixing angle. However, if the decays $\psi'\to\psi\pi\pi$, $\psi'\to\psi\eta$ both proceed through gluonic couplings ($\psi'\to\psi gg$, $gg\to\eta$ or $\pi\pi$), then the sizable coupling of η to gluons would allow the $\psi'\to\psi\eta$ decay to proceed without further inhibition.

(3) The results for the various F's can be used in order to shed some light on the microscopic structure of the η' meson. Consider the $q\bar{q}$ Bethe-Salpeter wave function for the neutral pseudoscalar P at zero $q\bar{q}$ separation[15]:

$$(2m_P)^{1/2}(\psi_P^{(q)})_{\alpha\beta}=\langle 0|q_\alpha^{+}(0)q_\beta(0)|P\rangle$$
$$\equiv -\tfrac{1}{4}i[\gamma_0(a_P^{(q)}\not{p}+b_P^{(q)})\gamma_5]_{\alpha\beta}.$$

It is straightforward to calculate a and b in terms of the F's for various assignments of $q=P$: e.g., $a_\eta^{(u)}=a_\eta^{(d)}=(\sqrt{3}/2)(\sqrt{2}F_{0\eta}+F_{8\eta})$, $b_\eta^{(u)}=b_\eta^{(d)}=(\sqrt{3}/4m_u)m_\eta^2(\sqrt{2}\tilde{F}_{0\eta}+F_{8\eta})$. One may then compute an average density $\rho\equiv{\rm tr}\psi^{+}\psi$ for $u(d)$ or s quarks in the rest frame of the π^0, η, and η' mesons. One finds for $m_u=m_d=8$ MeV, $m_s=160$ MeV, that

$$\rho_{\eta'}^{(u)}=\rho_{\eta'}^{(d)}=22F^{-3},\quad \rho_{\eta'}^{(s)}=0.24F^{-3},$$
$$\rho_\eta^{(u)}=\rho_\eta^{(d)}=2.2F^{-3},\quad \rho_\eta^{(s)}=0.87F^{-3},$$
$$\rho_{\pi^0}^{(u)}=\rho_{\pi^0}^{(d)}=1.5F^{-3}. \quad (26)$$

This points out an outstanding structural difference between the η' and either the π^0 or η: The probability of finding the light $q\bar{q}$ pair at small separation in the η' is about 10 times "normal" ("normal" $\sim 1F^{-3}\sim\rho_\eta^{(u)}\sim\rho_{\pi^0}^{(u)}$). Combined with the large value of $(A_{\eta'}/A_\eta)^2=10$, we see that the spatial extent of the light-quark–gluon part of the η' wave function is considerably smaller than that in the η and π^0.

To recapitulate *using neither SU(3) approximations nor soft-meson techniques* [except in obtaining Eq. (18)], I have found a set of parameters (F_π, F_K, $F_{8\eta}$, $F_{0\eta}$, A_η, $F_{8\eta'}$, $F_{0\eta'}$, $A_{\eta'}$) characterizing the physical pseudoscalar nonet which is consistent with known masses, with the Ward identities with $\mathcal{S}=0$,[16] and with the two-photon decays. The gluonic couplings $A_\eta, A_{\eta'}$ provide a consistent description of $\psi \to \eta(\eta')\gamma$ and may quantitatively explain the large rate for $\psi' \to \psi\eta$. The values obtained for the F's imply an anomalously small radius for the $u\bar{u}+d\bar{d}$ part of the η' wave function.

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Note added.—After the completion of this work, I received a preprint by V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zacharov [Institute for Theoretical Physics Report No. ITEP-73 (to be published)] which discusses $\psi \to \eta(\eta')\gamma$ from a standpoint related to that in the present work. Their estimates of A_η and $A_{\eta'}$ are made in the chiral limit, and hence an independent calculation of $\mathcal{S}$ is required.

[1]For a comprehensive review and list of references, see R. J. Crewther, CERN Report No. Ref. Th. 2546-CERN, 1978 (to be published).

[2]For earlier discussion on the existence of a fourth light isoscalar, see S. L. Glashow, in *Hadrons and Their Interactions*, Proceedings of the School of Physics "Ettore Majorana," 1967, edited by A. Zichichi (Academic, New York, 1968); S. Weinberg, Phys. Rev. D 11, 3583 (1975).

[3]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969). This anomaly was first introduced on phenomenological grounds by R. Arnowitt, M. Friedman, and P. Nath, Phys. Lett. 27B, 657 (1968).

[4]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976), and Phys. Rev. D 14, 3432 (1976).

[5]A. De Rújula, H. Georgi, and S. L. Glashow, Phys. Rev. D 12, 147 (1975); N. Isgur, Phys. Rev. D 12, 3770 (1975); N. H. Fuchs, Phys. Rev. D 14, 1912 (1976).

[6]E. Witten, Harvard University Report No. HUTP-79/A014 (to be published); G. Veneziano, CERN Report No. Ref. Th. 2651-CERN (to be published).

[7]For an earlier, unpublished version, see H. Goldberg, Northeastern University Report No. NUB 2397, May 1979. This work contains no reference to the (possible) QCD surface term, nor does it present the detailed analysis of the 2γ decays or the applications to ψ decays.

[8]Equations (13)–(15) without anomaly were first derived by R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. D 3, 594 (1971) [see Eqs. (C16)–(C18)]. The role of the anomaly was delineated by R. J. Crewther, Phys. Lett. 70B, 349 (1977).

[9]I use data on $\pi^0 \to 2\gamma$, $\eta \to 2\gamma$ from C. Bricman *et al.*, Phys. Lett. 75B, i, 1 (1978), and on $\eta' \to 2\gamma$ from G. S. Abrams *et al.*, Phys. Rev. Lett. 43, 477 (1979).

[10]By considering the scalar propagator $\Delta_\nu^{66} \equiv i\int \langle T \times \partial^\mu V_\mu^6(x)\partial^\nu V_\nu^6(0)\rangle$, $V_\mu^6 = \bar{q}\gamma_\mu(\lambda^6/2)q$, we may derive the additional Ward identity $\Delta_\nu^{66} = \delta = F_\kappa^2 m_\kappa^2 \geq 0$, where κ is the presumed 0^+ strange scalar at $\sim$1450 MeV, and $F_\kappa m_\kappa^2 \equiv \langle 0|\partial^\mu V_\mu^6|\kappa\rangle$. Hence, for my assignments (20), F_K is constrained to be $\leq 1.1F_\pi$. This is a bit low, and can be raised by raising $F_{8\eta}$. The principal results [Eq. (24)] are insensitive to such changes, and we postpone a full discussion to a future work.

[11]In the SU(3) limit ($F_\pi = F_K = F_{8\eta}$, $\delta = F_{0\eta} = 0$, $m_\pi = m_K = m_\eta$), Eqs. (13)–(15) require $F_{8\eta'} = A_\eta = 0$. Hence $A_\eta \neq 0$ is an SU(3) breaking effect. I would like to thank Professor K. Lane for an interesting discussion on this point.

[12]E. Bloom, in Proceedings of the SLAC Summer Institute, 1979 (unpublished). I would like to thank Dr. M. Ronan for communicating this result to me.

[13]If this model is to be trusted, it can also be used to bound $\mathcal{S}$, since $\mathcal{S}$ affects A_η and $A_{\eta'}$. E.g., for $\mathcal{S} \gtrsim 3(F_\pi m_\pi)^2 = (150\ \text{MeV})^4$, the ratio in Eq. (25) becomes ≤ 6.

[14]E.g., assume an average invariant mass of 400 MeV for the di-pion pair, giving an average c.m. momentum $\langle k_{\pi\pi}\rangle = 390$ MeV. The ratio $\Gamma(\psi' \to \psi\eta)/\Gamma(\psi' \to \psi\pi\pi) \sim (k_\eta R)^2 k_\eta/\langle k_{\pi\pi}\rangle \sim \frac{1}{10}$ for $R = 0.5$ F. A discussion of $\psi \to \psi\eta$ based on a sizable $c\bar{c}$ content for the η has been given by H. Harari, Phys. Lett. 60B, 172 (1976).

[15]It is the $a_P^{(q)}$ which control the $P \to 2\gamma$ decays.

[16]In the chiral limit, $\mathcal{S}$ controls the relation between $F_{\eta'}$ and $m_{\eta'}$. I do not consider this limit.