Gauge Wheel of Superfluid ^{4}He

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(Received 22 October 1979)

The assumption is explored that the equilibrium order parameter of a superfluid rotates with a rotating container. Since rotations and changes of phase are then intimately related in inhomogeneous configurations, effects like the ^{3}He-A "gauge wheel" of Liu and Cross should arise in any superfluid. Such effects may be difficult to observe, but it is shown that their existence in superfluid ^{4}He is directly implied by nothing more than the conventional equations of two-fluid hydrodynamics.

Liu and Cross[1] have pointed out that suitable rotations of immersed wheels or surrounding walls can induce unusual flow and thermal effects ("gauge-wheel effects") in the A phase of superfluid helium-3. The equivalence of a rotation about the gap anisotropy axis to a change in phase of the order parameter plays a central role in their argument, suggesting that gauge wheel effects are characteristic of the unique type of superfluidity possessed by ^{3}He-A. This is not, in fact, the case. The primary purpose of this Letter is to show that gauge-wheel effects quite similar to those discussed by Liu and Cross also occur in superfluid ^{4}He. Gauge-wheel effects are not peculiar to ^{3}He-A, though the special form of the A-phase order parameter lends to them (and other superfluid properties) an unusual richness and diversity of form

What *is* essential for guage-wheel is that the order parameter of a superfluid should rotate (or move) with the container when the fluid is in equilibrium with a rotating (or moving) vessel. The existence of such rotating equilibrium has recently been disputed on several grounds.[2,3] We believe none of these criticisms to be well founded,[4] and a secondary purpose of this Letter is to emphasize that nontrivial consequences of the assumption of rotating equilibrium in ^{4}He are entirely contained in the conventional two-fluid hydrodynamics of Landau.[5] Therefore in what follows we shall use the assumption of rotating equilibrium only for its (considerable) heuristic power, showing that *if* the order parameter in ^{4}He does move with the container, then gauge-wheel effects arise in many simple geometries. We shall then set aside the assumption and derive one such effect using only two-fluid hydrodynamics.

To understand gauge-wheel effects in ^{4}He it is essential to distinguish the phase of the superfluid order parameter from its gradient $\vec{v}_s$, since the irrotational character of $\vec{v}_s$ has nothing to do with whether or not the phase rotates. The assumption of rotating equilibrium asserts that where superfluid ^{4}He is in equilibrium with the walls of a container rotating with angular frequency ω about an axis $\hat{z}$, the time dependence of the phase φ is given by

$$\varphi(\vec{r},t) = -\mu t m/\hbar + \varphi(R^{-1}(\hat{z},\omega t)\vec{r},0), \qquad (1)$$

where μ is the chemical potential and R is a rotation.

The consequences of this assumption for the equilibrium of a rigidly rotating container are simple and relatively uninteresting: In ^{3}He-A the texture in the anisotropy axis $\hat{l}$ rotates with the container, and in both superfluid ^{3}He-A and superfluid ^{4}He the vector field $\vec{v}_s$ undergoes rigid-body rotation. [This rotating vector field is, of course, the gradient of a rotating scalar field and it is therefore, as required, irrotational (i.e., curl free).] Note that a velocity field with axial symmetry is invariant under such a rotation, so that rotating equilibrium is compatabile with the familiar behavior of vortex-free rotating ^{4}He. If a vortex lattice is present $\vec{v}_s$ will not have axial symmetry, and the observed rotation of the vortex lattice with the container[6] can be viewed as a direct confirmation of Eq. (1).

More interesting things—the gauge-wheel effects—happen when various parts of the surface of the container rotate at different rates, or move with different velocities. If the order parameter remains in local equilibrium with the nearby moving wall, then it can undergo distortions as well as rigid-body rotations. In ^{3}He-A, because the order parameter has a vector structure, these distortions can build up even when the initial configuration of the order parameter is entirely uniform. In ^{4}He, however, the order parameter is a scalar phase, which must be nonuniform if its motion with the wall is to alter its form.[7] The nonuniformity can be quite simple: The presence of a persistent current in the initial configuration is enough to produce gauge-wheel effects.

As a simple example of this, consider two long, coaxial circular cylinders, the space between which is filled with superfluid ^{4}He, the necessary nonuniformity in the order parameter being provided by a nonvanishing circulation in $\vec{v}_s$ about the common axis $\hat{z}$ of the cylinders.[8] When both cylinders are stationary and the superfluid is in equilibrium for the given circulation, then at any instant of time the phase of the order parameter is a monotonically increasing (multivalued) function of angle about the cylinder axis. If the phase is plotted along the z axis over any cross-sectional plane perpendicular to $\hat{z}$, the resulting graph is a helical ramp which intersects each cylinder in helices of identical pitch (see Fig. 1). As time evolves the helices move uniformly along the z direction at a rate determined by the uniform chemical potential [see Eq. (1)].

When both cylinders rotate together at the same rate, then the assumption of rotating equilibrium requires both helices to move with the walls, thereby superposing on the uniform motion of the graph of φ determined by the chemical potential the additional uniform motion characteristic of the spiral on a rotating barber pole. Note that the gradient of the phase, and hence the superfluid velocity $\vec{v}_s$, is completely unaffected by such a rigid motion of the entire helical ramp: Rotating equilibrium is fully compatible with the refusal of a uniform supercurrent to participate in the motion.

Gauge-wheel effects arise if only one of the cylinders is set into rotation. The additional "barber-pole" contribution to the rate of change of the phase will then initially be communicated only to the helium in the neighborhood of the moving cylinder. A radial phase difference will thus begin to grow between the two cylinders, giving rise to a growing radial component of $\vec{v}_s$. This radial superflow will continue to increase until a steady state is reached in which the effect of the different rates of rotation of the cylinders on the radial phase difference is balanced by the effect of a rotationally induced radial chemical potential gradient.[8] In the steady state the radial $\vec{v}_s$ will be accompanied by an oppositely directed normal-

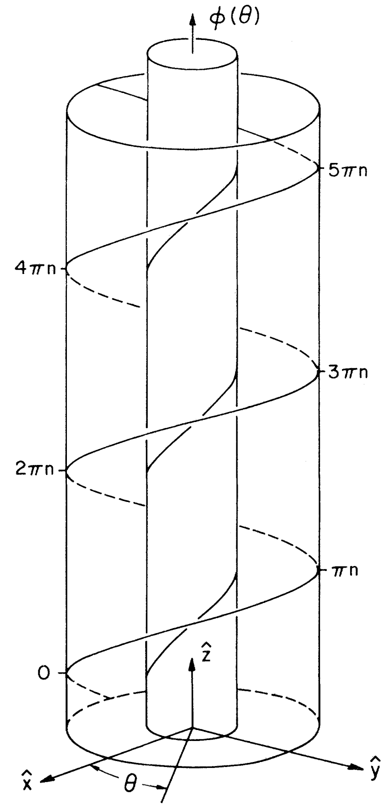

FIG. 1. The phase in helium-4 when an azimuthal supercurrent flows between two stationary cylindrical walls. The phase φ is a multivalued function of θ, and independent of r and z. It is plotted along the z axis as a function of θ. The resulting graph intersects the walls in helices of identical pitch proportional to the magnitude of $\vec{v}_s$. If the outer cylinder is now set into rotation (in the sense of increasing θ) then the graph of φ on the outer cylinder will move in the z direction as the order parameter moves with the wall, leading to an inwardly directed radial phase gradient. (Ignore the uniform motion along $\hat{z}$ at a rate $\mu/\hbar$ of the entire graph.)

fluid velocity $\vec{v}_n$ to ensure that there is no net radial mass flow, and there will therefore be a temperature difference between the cylinders. In the linear regime the size of the steady-state radial velocity fields and the size of the temperature drop will be proportional to the initial rate of relative phase change. This in turn is proportional to the product of the circumferential speed ω of the rotating cylinder with the change in phase per cycle which is proportional to the circulation of $\vec{v}_s$ around the z axis.[9]

Precisely this effect is implied by conventional two-fluid hydrodynamics. For simplicity we replace the concentric cylinders by plane walls perpendicular to the x axis at $x=0$ and $x=d$. The quantized circulation is provided by a y component of $\vec{v}_s$ that is independent of position and constrained to retain the value v throughout the approach to the steady state. We consider a geometry in which the wall at $x=0$ is stationary while the wall at $x=d$ moves in the y direction with constant speed w. We seek a steady-state solution to the hydrodynamic equations in which all quantities vary with position only in the direction ($\hat{x}$) perpendicular to the walls. The wall motion gives the boundary conditions $v_{ny}(0)=0$, $v_{ny}(d)=w$. The absence of a transverse mass flow imposes a second condition, $v_{sx}=-(\rho_n/\rho_s)v_{nx}$. A third boundary condition is required to insure that any resulting steady-state counterflow is internally generated and not externally imposed. This can be done by insisting on a thermally insulating wall at $x=0$:

$$j^{\text{thermal}}(0)=Tsv_{nx}(0)-KT'(0)=0,$$

where s is the entropy density, K the thermal conductivity, and $T'=dT/dx$.[10]

We solve the hydrodynamic equations[11] when the wall velocity w is small compared with the superfluid velocity v parallel to the walls. If the equations are linearized in w/v then there is no spatial dependence to s, ρ_n, ρ_s, and the dissipative coefficients to leading order. It is easily verified that the linearized equations have a solution in which v_{nx}, v_{sx}, the pressure P, and the chemical potential μ_s are all constant.[12] Because P and μ_s are uniform, the Gibbs-Duhem equation reduces to

$$T'=\rho_n v v_{ny}'/s, \tag{2}$$

which gives a temperature drop of the expected form:

$$\Delta T = T(\alpha) - T(0) = (\rho_n/s)vw. \tag{3}$$

The magnitude of the transverse normal flow v_{nx} is related to the temperature gradient by the rate equation for the entropy, which requires $(sv_{nx}-KT'/T)'$ to be equal to the entropy production per unit volume. Since the entropy production is of second order, this together with the boundary condition on the heat current relates v_{nx} to the transverse temperature gradient:

$$v_{nx}=KT'/Ts. \tag{4}$$

The y component of the rate equation for the momentum density is solved by $v_{ny}=wx/d$, which with (2) and (4) gives

$$v_{nx}=(K\rho_n/Ts^2 d)vw. \tag{5}$$

The assumption that all velocities are small compared with v leads to the condition $K\rho_n w/Ts^2 d \ll 1$, for the validity of our analysis. A second condition is given by the rate equation for y momentum, where nonlinear terms will, in fact, be small provided that $\rho_n v_{nx} d/\eta \ll 1$, where η is the shear viscosity. With Eq. (5), this condition becomes $(K/\eta T)(\rho_n/s)^2 vw \ll 1$. The crudest order-of-magnitude estimates [$Ts\sim\rho_n c^2$, $K\sim sc^2\tau$, $\eta\sim\rho_n c^2\tau$, where c is the velocity of (first or second) sound and τ is a relaxation time] reduce these conditions to the modest requirements $(w/c)(c\tau/d)\ll 1$ and $wv\ll c^2$. They also make ΔT unobservably small.[13]

We shall not enter here into a discussion of possible dynamical gauge-wheel effects in ^{4}He. We offer our steady-state gauge wheel of helium-4 as a thought experiment designed to show that the gauge wheel is not peculiar to ^{3}He-A, and to show that the rotation of the superfluid order parameter with the walls of the container is implicit in the two-fluid hydrodynamics of helium-4 given by Landau almost 40 years ago.

We are grateful to R. Combescot and K. Nagai for sending us preprints, and would like to thank Eric Siggia for several helpful conversations. This work is supported by the National Science Foundation under Grants No. DMR 76-24011 and No. DMR 77-18329.

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[1]Mario Liu and M. C. Cross, Phys. Rev. Lett. 43, 296 (1979). Our use of their term "gauge wheel" is based on the following criterion: A system is capable of gauge-wheel effects provided there is some geometry in which the effect on the order parameter of a rotation can be completely undone by a gauge transformation.

[2]R. Combescot, to be published; R. Combescot and T. Dombre, to be published.

[3]K. Nagai, to be published.

[4]We shall give a detailed justification of the assumption of rotating equilibrium in a forthcoming paper.

[5]We show in Ref. 4 that the assumption of rotating equilibrium is, in fact, implicit in the derivation of two-fluid hydrodynamics.

[6]E. J. Yarmchuk *et al.*, Phys. Rev. Lett. 43, 214 (1979). (Note that the "photographs" were taken with a "camera" that rotated with the helium.)

[7]This difference between the two superfluids is reflected in the way in which the gauge-wheel effects manifest themselves in the hydrodynamic equations. In both superfluids a central role is played by a term that

is sometimes overlooked, when insufficient care is taken in specifying the variable that is to be identified with the chemical potential. This point is discussed in Ref. 4. In addition, because of the vector character of the ^{3}He-*A* order parameter, there is an explicit term proportional to $\vec{l}\cdot\nabla\times\vec{v}_n$ in the equation governing the time rate of change of the effective phase. This additional term is strongly emphasized in Ref. 1, but the first term is also implicit in their discussion and can produce fully comparable effects.

[8]We emphasize that this circulation (which is the hydrodynamical manifestation of the crucial nonuniformity in the initial order parameter) is essential for the appearance of the gauge-wheel effect. The circulation distinguishes this geometry from ordinary superfluid couette-flow experiments, which are not necessarily carried out against a fixed background supercurrent.

[9]The same effect will take place in the *A* phase of superfluid helium-3 in the presence of a circulating supercurrent, even when the anisotropy axis is radial.

[10]In a nonlinear analysis the heat current at the opposite wall is determined by the bulk entropy production. In a linear analysis the entropy production is second order and the thermal current can be taken to vanish for all x.

[11]We use these in the form given by I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), p. 66.

[12]We use the symbol μ_s to denote what Khalatnikov denotes by the symbol μ. Since whatever name one gives it, the quantity is determined by the hydrodynamics, this change in nomenclature has no consequence for our analysis. However we shall show in Ref. 4 that what is conventionally called the chemical potential in many treatments of superfluid hydrodynamics is, in fact, the chemical potential in the local rest frame of the superfluid μ_s. This differs from the true chemical potential μ by terms of second order in v_n and v_s ($\mu=\mu_s+\frac{1}{2}v_s^2-\vec{v}_n\cdot\vec{v}_s$) and for many purposes μ and μ_s can be identified. In the case of gauge-wheel effects the second-order terms are of crucial importance, and one loses considerable insight into the underlying physics by confusing μ_s with μ. In particular, the fact that the two-fluid hydrodynamics gives a constant μ_s is not incompatible with our earlier assertion that the phase winding is balanced by a (true) chemical-potential gradient in the steady state.

[13]This does not necessarily imply that the temperature drop remains unmeasurably small as w increases, though when $w/v\approx 1$, the fluid may respond through vortex nucleation rather than by producing a chemical potential (and hence temperature) drop. Note that from this point of view gauge-wheel effects are cleaner in ^{4}He than in ^{3}He-*A*, since the possibility for similar complications due to textural motions is absent in the more common superfluid.

Third-Sound Velocity and Onset on Grafoil

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(Received 26 October 1979)

Third-sound velocity and onset measurements have been made for ^{4}He adsorbed on Grafoil foam. The third-sound velocity has a dependence on film thickness and temperature similar to that of third sound on glass substrates, with some distinguishing features. The film vapor pressure at onset has, to some degree, the universal temperature dependence found for other substrates. The superfluid areal density divided by the onset temperature agrees with the universal constant predicted by the Kosterlitz-Thouless-Nelson theory to within 11%.

Recently there has been much interest in the superfluid transition in adsorbed ^{4}He films due to the successful predictions of the Kosterlitz-Thouless-Nelson (KTN) theory.[1] Third-sound results[2] and Andronikashvili pendulum results[3] are in very good agreement with the prediction that at the onset of superfluidity

$$(\bar{\rho}_s d/T)_{\text{onset}}=(2/\pi)(m/\hbar)^2k_B, \tag{1}$$

where $\bar{\rho}_s$ is the average superfluid volume density in the film of thickness d at a temperature T, and m, $\hbar$, and k_B are the helium mass, Planck's constant, and Boltzmann's constant, respectively. However, there has been some speculation as to the applicability[4-6] of the KTN theory to helium adsorbed on the graphite substrates Grafoil, ZYX, and Grafoil foam.[7] There are a number of different experiments involving various substrates, including Grafoil, which give to some degree a universal temperature dependence of film vapor pressure at onset. There is one published set of Grafoil mass-flow data[6] which seems to deviate significantly from the universal behavior, and some recent unpublished data[8-9] give in-