particular for the quadrupole-quadrupole form considered here. It will generally have a minimum for some value of β and γ, for given ϵ, κ, and χ. For $\chi<0$, the minimum always occurs at $\gamma=0$, while for $\chi>0$ it occurs at $\gamma=\pi/3$. When $\chi=0$ the energy surface is independent of γ, as appropriate to a γ-unstable rotor, and has a minimum at $\beta\cong 1$ (β approaches unity for large N). For the SU(3) limit, with $\epsilon=0$ and $\chi=-\frac{1}{2}\sqrt{7}$, the minimum occurs at $\gamma=0$, $\beta\cong\sqrt{2}$ (β approaches $\sqrt{2}$ for large N). In the SU(5) limit, with $\kappa=0$, the energy surface is again independent of γ, with a minimum at $\beta=0$, as appropriate to a vibrator. Thus the behavior of the energy surface exactly parallels that derived from our collective Hamiltonian above.[3]

In conclusion, we have shown that by means of the intrinsic state given in (2) we can convert the IBM Hamiltonian into a differential operator in terms of the shape parameters β,γ, and Euler angles of an ellipsoid. The rotational degrees of freedom are completely decoupled from the intrinsic variables, and the moments of inertia have mass parameters which depend on the shape variables, but are similar to those given by Bohr.[2] However, the intrinsic Hamiltonian has a complicated dependence on the shape parameters including coupling between the β and γ shape variables. In the small-β and -γ limit the Hamiltonian derived from the boson Hamiltonian reduces to the Bohr Hamiltonian. We have also shown that the derived collective Hamiltonian has the correct behavior in each of the well-known limiting cases of the IBM.

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[(a)]On leave from the Weizmann Institute of Science, Rehovot, Israel.

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Classical Limit of the Interacting-Boson Model

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A set of classical variables associated with the interacting-boson model is defined. It is shown that these variables can be put in one-to-one correspondence with Bohr's liquid-drop variables. The classical equilibrium "shapes" corresponding to the three limits of the interacting-boson model are analyzed and the nature of the "shape" phase transitions between them is studied.

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The classical limit of quantum systems is one of the oldest problems in quantum mechanics. This problem appears whenever one formulates a theory in terms of quantum (particle) variables and wishes to interpret it in terms of classical (geometrical) variables. The correspondence between classical and quantum variables is in general ambiguous. However, Gilmore[1] and others have recently shown that an unambiguous definition can be given whenever the operators X of which one wants to find the classical limit belong to any compact Lie algebra G, as for example, the unitary algebra U(r). We have applied this new and powerful mathematical method to the study of the classical limit of the interacting-boson model.[2] As a result, we are able to show that the Bohr-Mottelson liquid-drop model[3] appears (up to a homomorphism) as the classical limit of the interacting-boson model. Thus our study bridges the gap between these two models

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which have been widely used for the description of collective states in nuclei.

To begin with, we note that in the interacting-boson model one describes an even-even nucleus as a system of N bosons able to occupy two levels, one with $L=0$ (s boson) and one with $L=2$ (d boson). Denoting by $b_i^\dagger$ (b_i) ($i=1,\ldots,6$) the creation (annihilation) operators for bosons ($b_1 \equiv s$, $b_{2,\ldots,6} \equiv d$), it is easy to see that the 36 operators $G_{ii'} = b_i^\dagger b_{i'}$ close under the Lie algebra of U(6). Thus, this problem satisfies Gilmore's criterion, with $r=6$. Although, with the algorithm of Gilmore *et al.*, it is possible to study a large number of classical properties of this quantum system, in this letter we limit ourselves to: (i) define (up to a homomorphism) classical (geometrical) variables appropriate to the interacting boson model, thus associating a "shape" to it; (ii) construct, with these variables, an upper bound to the total ground-state energy which converges to the exact classical value when $N \to \infty$, and (iii) study the nature of possible phase transitions. The construction of the $\mathfrak{D}$ algebra, differential operator, realization of the boson operators $b_i^\dagger, b_i$, which is equally possible, as shown in Ref. 4, will not be discussed in the present Letter.

The first question one has to answer is how many classical variables are needed. Gilmore, Bowden, and Narducci[4] have shown that for a quantum system of N bosons described by the group U(r), one needs $r-1$ classical complex variables, the remaining complex variable being eliminated by insisting that one must remain within the totally symmetric representation $[N]$ of U(r) (conservation of boson number). Thus, for U(6), one needs *five* complex variables. For these one can choose, in principle, any convenient parametrization one wishes. From the group theoretical point of view, the most natural choice is to use the variables associated with the coset space $U(r)/U(r-1)\otimes U(1)$ ($r=6$). This space, labeled by coordinates x_u ($u=2,\ldots,6$), may be identified with the five-dimensional complex sphere, the sixth coordinate x_1 being related to the others by $x_1^2 + \sum_{u=2}^6 x_u^* x_u = 1$. [Instead of five complex x_i's, ten angles θ_i, φ_i ($i=2,\ldots,6$) may also be used.] Furthermore, it can be shown that, for the study of static properties, the five coordinates may be chosen to be real ($\varphi_i=0$, $i=2,\ldots,6$).[5] We find it convenient to introduce the set of five real coordinates α_μ ($\mu=-2,-1,0,+1,+2$). While the coordinates x_u (or θ_u, φ_u) are in one-to-one correspondence with atomic coherent quantum states, the coordinates we introduce are in one-to-one correspondence with quantum states of the form $(s^\dagger + \sum_{u=-2}^{+2} \alpha_\mu d_\mu^\dagger)^N|0\rangle$, which we will denote by $|N,\underline{\alpha}\rangle$ in the following. The coordinates $\underline{\alpha}$ are related to Gilmore's coordinates θ_u by a stereographic projection from the "five-dimensional" Bloch sphere onto a plane tangent to it. Thus, in conclusion, the geometry of the interacting-boson model is that of a five-dimensional space. If one wishes, one may visualize this geometry by associating to each point $\underline{\alpha}$, a point on the surface of a deformed body with radius $R/R_0 = 1 + \sum_{\mu=-2}^{+2} \alpha_\mu Y_{2\mu}(\theta,\varphi)$. The five variables α_μ can then be replaced by two intrinsic variables β, γ and three Euler angles θ_1, θ_2, θ_3, which determine the orientation in space of the deformed body (Bohr variables[3]). All these variables are equivalent, up to a homomorphism. This establishes the correspondence between the interacting-boson model and the liquid-drop model.

The next step is to provide, with the classical variables introduced above, an upper bound to the ground-state energy of the system, which, for $N \to \infty$, converges to the exact energy. This is simply done by constructing the energy functional

$$E(N,\underline{\alpha}) = \langle N,\underline{\alpha}|H|N,\underline{\alpha}\rangle / \langle N,\underline{\alpha}|N,\underline{\alpha}\rangle, \qquad (1)$$

and minimizing it with respect to $\underline{\alpha}$, $\delta E(N,\underline{\alpha})/\delta\underline{\alpha} = 0$. This algorithm provides a rigorous definition of the equilibrium "shape" associated with a given boson Hamiltonian H and boson number N, which has the property of converging to the classical equilibrium "shape" when $N \to \infty$. We remark incidentally that the new techniques developed by Gilmore and Feng[5] also allow one to construct a lower bound which converges to the exact energy when $N \to \infty$.

We are now in a position to calculate the equilibrium "shapes" associated with the three limits of the interacting boson model, (I) U(5),[6] (II) U(3),[7] and (III) O(6)[8]. This problem is of particular importance since these three limits have been observed experimentally. These three limits, associated with dynamical symmetries of the Hamiltonian H, appear as shape *phases* at the classical level ($N \to \infty$). The technique described here allows one also to give an algebraic description of the nature of the transition between one phase and another. In order to calculate the equilibrium shape of the three limits (I), (II), and (III), it is sufficient to consider the energy functional $E(N,\underline{\alpha})$ associated with some of the Casimir invariants of the respective group chains. For the limit (I) we take[6] $H^{(\mathrm{I})}=\epsilon n_d$, for limit (II) we take[7] $H^{(\mathrm{II})} = -\kappa[C_{\mathrm{SU}(3)} - 4N^2 - 6N]$, where $C_{\mathrm{SU}(3)}$ is the quadratic Casimir operator of SU(3) and for the limit (III) we take[8] $H^{(\mathrm{III})}=\kappa' P_6$. The corresponding energy functionals in terms of Bohr variables are given by

$$E^{(\mathrm{I})}(\beta,\gamma)=\epsilon N[\beta^2/(1+\beta^2)],$$
$$E^{(\mathrm{II})}(\beta,\gamma)=\kappa N(N-1)\left(\frac{1+\frac{3}{4}\beta^4-\sqrt{2}\,\beta^3\cos 3\gamma}{(1+\beta^2)^2}\right), \quad (2)$$
$$E^{(\mathrm{III})}(\beta,\gamma)=\kappa' N(N-1)[(1-\beta^2)/(1+\beta^2)]^2,$$

where only the values ϵ, κ, $\kappa' \geqslant 0$ are realistic. By minimizing $E(\beta,\gamma)$ with respect to β and γ, one finds that phases (I) and (III) are γ independent and have minima at $\beta=0$ (phase I) and $\beta=1$ (phase III). Phase (II) has a sharp minimum at $\gamma=0°$ (axial symmetry) and $\beta=\sqrt{2}$. It is interesting to note that if we reverse the sign of $\chi=-\sqrt{7}/2$ in the operator $Q=(d^\dagger\times s+s^\dagger\times\tilde{d})^{(2)}+\chi(d^\dagger\times\tilde{d})^{(2)}$, the minimum appears at $\gamma=60°$, $\beta=\sqrt{2}$, in agreement with the reversal of sign of the quadrupole moments in the quantum version. (We assume $\beta \geqslant 0$, $0° \leqslant \gamma \leqslant 60°$, as usual.)

In addition to providing a straightforward connection with the geometrical description, the algorithm discussed here allows one to study in a rigorous way the nature of shape phase transitions in finite nuclei. To this end,[1,5] one first constructs the classical limit ($N\to\infty$) of some appropriate combination of (2), then studies the nature of the discontinuity at the critical point, and, finally, converts this statement back to the case of finite N where the phase transition does not appear as a discontinuity but rather as a more (N large) or less (N small) abrupt change in observable quantities. For example, if one wants to study the nature of the phase transition between limits (I) and (III), one starts from the Hamiltonian $H^{(\mathrm{I}+\mathrm{III})}=H^{(\mathrm{I})}+H^{(\mathrm{III})}$. Equation (2) the gives for the energy per particle, when $N\to\infty$,

$$\mathcal{E}^{(\mathrm{I}+\mathrm{III})}(\beta,\gamma)=\frac{\beta^2}{1+\beta^2}+\eta'\left(\frac{1-\beta^2}{1+\beta^2}\right)^2, \quad (3)$$

where $\mathcal{E}=E/\epsilon N$, $\eta'=\kappa'(N-1)/\epsilon$. As a function of the coupling parameter η', the value of β for which ϵ is at a minimum, $E_{\min}$, shifts from $\beta=0$ (spherical limit, $\eta'\leqslant\frac{1}{4}$) to $\beta=[(1-4\eta')/(1+4\eta')]^{1/2}$ (deformed limit, $\eta'\geqslant\frac{1}{4}$). A study of $\mathcal{E}_{\min}$ and its derivatives at the critical point $\eta'=\eta_c'$ determines the nature of the phase transition.[1,5] This study gives the following results: (a) The transition from limit (I) to (II) is a first-order phase transition; that is, at the critical value $\eta=\eta_c=\frac{4}{9}$ [$\eta=\kappa(N-1)/\epsilon$], $\delta\mathcal{E}_{\min}/\delta\eta$ is discontinuous; (b) the transition from limit (I) to (III) is a second-order phase transition; that is, at the critical value $\eta'=\eta_c'=\frac{1}{4}$, $\delta^2\mathcal{E}_{\min}/\delta\eta'^2$ is discontinuous; (c) between limits (II) and (III) no phase transition occurs for physical values of the coupling parameters κ, $\kappa'\geqslant 0$. (A phase transition does occur for unphysical values $\eta''=\kappa'/\kappa<0$.)

The possibility to study phase transitions in finite systems in a rigorous way opens a new per-

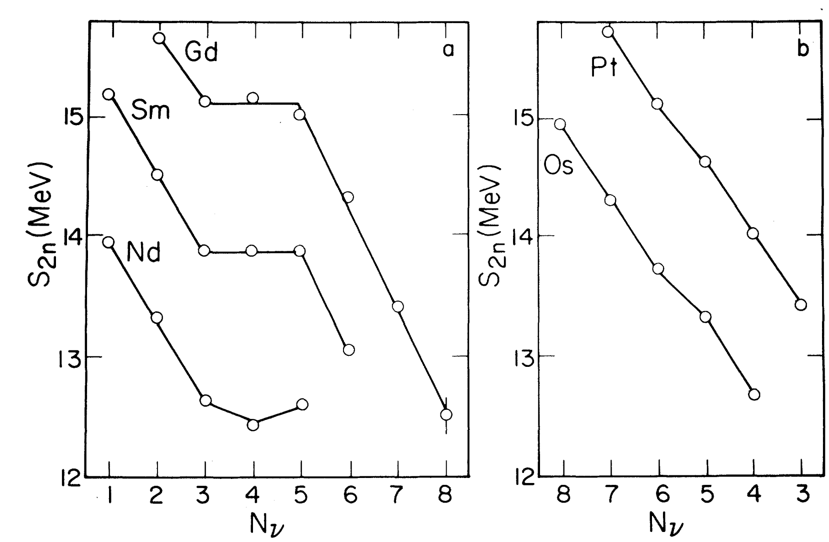

FIG. 1. Two-neutron separation energies, (a) S_{2n}, in the Nd-Sm-Gd isotopes and (b) in the Os-Pt isotopes, as a function of the number of neutron bosons, N_ν.

spective into the analysis of experimental data for phase transitions in nuclei. For example, the Sm and Gd isotopes show a distinct discontinuity in the two-neutron separation energies, S_{2n}, whereas the same quantity in the Os and Pt isotopes is smooth (Fig. 1). In the interacting boson model, the former case is an example of a transition from symmetry (I) to (II),[9] while the latter is an example of a transition from (II) to (III).[10] Since S_{2n} may be related[9] to the derivative of the ground-state energy with respect to $\eta=\kappa(N-1)/\epsilon$ (or η'), the results of the present paper agree well with the experimental situation.

In conclusion, we have been able, using Gilmore's algorithm, to (i) associate classical shape variables to the interacting boson model; (ii) construct an upper bound to the ground-state energy which converges to the exact value when $N\to\infty$, and (iii) study the nature of shape-phase transitions in nuclei. A full account of the details of the derivation of the various results presented here will be published elsewhere.[11]

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Note added—After this work was completed we learned that a similar investigation has been performed by J. N. Ginocchio and M. W. Kirson.[12]

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