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Comment on Chiral and Conformal Anomalies

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It is shown that the trace anomaly is also identified with the Jacobian factor for the functional measure under the conformal transformation in the path-integral formalism. The path-integral formulation of anomalous Ward-Takahashi identities is then translated into a simple algebraic characterization of chiral and conformal anomalies. This exemplifies some of the common features shared by the topological and nontopological anomalies.

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The anomalous chiral Ward-Takahashi (W-T) identities[1] (Adler-Bell-Jackiw anomaly) can be formulated in a simple manner with the aid of the index theorem[2] in the path-integral formalism.[3] It was also recognized that the chiral anomaly is related to the fact that the covariant "energy" operator $\not{D}$ and γ_5 cannot be simultaneously diagonalized.[4] In the present note I show that the trace anomaly[5] is also formulated as the Jacobian factor arising from the conformal transformation in the functional measure.[6] I then translate the path-integral formulation of anomalous W-T identities into a simple algebraic characterization of chiral and conformal anomalies.

I start with the simplest conformal-invariant theory of a "free" scalar field in the background gravitation field (R is the scalar curvature):

$$S \equiv \tfrac{1}{2}\int [\, g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \tfrac{1}{6}R\varphi(x)\varphi(x)](\sqrt{-g})d^4x \quad (1)$$

suitably continued to the Euclidean space. The Wick rotation is performed in the *local Lorentz frame*[7] as $h_0{}^\mu \to +ih_4{}^\mu$ and $\sqrt{-g} \equiv \det h_\mu{}^a \to -i\sqrt{g} \equiv -i\det h_\mu{}^a$, where $h_a{}^\mu(x)$ is the tetrad satisfying $h_a{}^\mu(x)h^{\nu a}(x) \equiv g^{\mu\nu}(x)$.

As is well known, (1) is invariant under the local conformal transformation

$$g^{\mu\nu}(x) \to e^{2\alpha(x)}g^{\mu\nu}(x) \quad [g_{\mu\nu}(x) \to e^{-2\alpha(x)}g_{\mu\nu}(x)],$$
$$\varphi(x) \to e^{\alpha(x)}\varphi(x), \quad (2)$$

without the use of the equation of motion. This property is compactly specified by

$$\frac{\delta_g}{\delta\alpha(x)} S + \frac{\delta_\varphi}{\delta\alpha(x)} S \equiv 0. \quad (3)$$

I next define the scalar density field by

$$\tilde{\varphi}(x) \equiv [g(x)]^{1/4} \varphi(x) \quad (4)$$

which transforms under (2) as

$$\tilde{\varphi}(x) \to e^{-\alpha(x)} \tilde{\varphi}(x). \quad (5)$$

The action $\tilde{S}(g, \tilde{\varphi}) \equiv S(g, \varphi)$ satisfies the relation

$$\frac{\delta_g}{\delta\alpha(x)} \tilde{S} + \frac{\delta_{\tilde{\varphi}}}{\delta\alpha(x)} \tilde{S} \equiv 0 \quad (6)$$

instead of (3). The Euclidean path integral in the background gravitational field is defined by

$$Z(g^{\mu\nu}, J) = N^{-1} \int d\mu \exp\{\int d^4x [\tfrac{1}{2} g^{1/2}(g^{\mu\nu} \partial_\mu\varphi \partial_\nu\varphi + \tfrac{1}{6} R\varphi^2) + i\,\tilde{\varphi}(x) J(x)]\}. \quad (7)$$

After a change of the basis from the x representation to the n representation,

$$\varphi(x) \equiv \textstyle\sum_n a_n \varphi_n(x) = \sum_n \langle x|n\rangle a_n \quad (8)$$

with a complete orthonormal set belonging to the *Hermitian* "energy" operator[7]

$$(D_\mu D^\mu - R/6)\varphi_n(x) = \lambda_n \varphi_n(x), \quad \int \varphi_n(x)\varphi_m(x) g^{1/2}\, dx = \delta_{n,m}, \quad (9)$$

the path-integral measure with a weight[6] $\frac{1}{2}$ is given by

$$d\mu \equiv \textstyle\prod_x \mathfrak{D}\{[g]^{1/4}\varphi(x)\} = \prod_n da_n. \quad (10)$$

The transformations (8) and (10) are formally unitary and the coefficients a_n are *ordinary* numbers. Because of this manifestly covariant definition of the measure, the basic integration variable inside the action is given by $\tilde{\varphi}$ in (4), and the action $\tilde{S}$ satisfies the identity (6). We thus obtain

$$\frac{\delta_g}{\delta\alpha(x)} Z(g^{\mu\nu}, J)\Big|_{\alpha=0} \equiv \langle T_{\mu\nu}(x)\rangle g^{\mu\nu}(x)\sqrt{g} = \left\langle \frac{\delta_g}{\delta\alpha(x)} \tilde{S}\right\rangle\Big|_{\alpha=0} = -\left\langle \frac{\delta_{\tilde{\varphi}}}{\delta\alpha(x)} \tilde{S}\right\rangle\Big|_{\alpha=0}. \quad (11)$$

The use of $\tilde{\varphi}$ also shows that the energy-momentum tensor generated by the quantized matter field $\langle T_{\mu\nu}(x)\rangle$ in (11) differs from the naive expectation value of the classical energy-momentum tensor defined by $\delta S/\delta g^{\mu\nu}(x)$ with the original action S of Eq. (1); this property is partly responsible for the trace anomaly generated by the quantized gauge field such as the electromagnetic field,[5] although the gauge field itself is scalar under the conformal transformation (2).

I next derive the local W-T identity arising from the transformation of the integration variable (5). Under this change of the variable, we have

$$\tilde{\varphi}'(x) \equiv e^{-\alpha(x)}\tilde{\varphi}(x) = \textstyle\sum_n a_n e^{-\alpha(x)} g^{1/4} \varphi_n(x) \equiv \sum_n a_n' g^{1/4}\varphi_n(x); \quad (12)$$

thus we obtain

$$a_n' = \textstyle\sum_m \int \varphi_n(x) e^{-\alpha(x)} \varphi_m(x)\sqrt{g}\, dx\, a_m. \quad (13)$$

The Jacobian factor for infinitesimal $\alpha(x)$ is then given by

$$d\mu' = \exp[-\textstyle\sum_n \int \alpha(x)\varphi_n(x)\varphi_n(x) g^{1/2} dx]\, d\mu. \quad (14)$$

The W-T identity is a statement that the variation of the action $\tilde{S}$ and the variation of the measure exactly cancel, namely, $\delta_{\tilde{\varphi}} Z/\delta\alpha(x) \equiv 0$. We thus obtain the conformal W-T identity [by discarding the source term in (7) for simplicity]

$$\left\langle \frac{\delta_{\tilde{\varphi}}}{\delta\alpha(x)} \tilde{S}\right\rangle\Big|_{\alpha=0} = A_W(x)\sqrt{g} \quad (15)$$

with

$$A_W(x) \equiv \textstyle\sum_n \varphi_n(x)\varphi_n(x). \quad (16)$$

Combining (11) and (15), we have the *bare* form of the trace identity:

$$g^{\mu\nu}(x)\langle T_{\mu\nu}(x)\rangle = -A_W(x). \quad (17)$$

The anomaly factor (16) may be evaluated by summing the series starting from the small eigenvalues λ_n in (7):

$$A_W(x) = \lim_{M\to\infty} \sum_n \varphi_n(x) \exp(-\lambda_n/M^2)\varphi_n(x) = \lim_{M\to\infty} \left\{ \frac{M^4}{(4\pi)^2} + \frac{1}{2880\pi^2}\left[R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + D^\mu D_\mu R\right] \right\}, \tag{18}$$

where the well-known result appearing in the intermediate stage of the ζ regularization[8] is used. If one renormalizes the relation (17) in the flat space-time limit, one recovers the anomalous trace identity[5,6]

$$g^{\mu\nu}(x)\langle T_{\mu\nu}(x)\rangle_{\rm ren} = \frac{(-1)}{2880\pi^2}\left[R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + D^\mu D_\mu R\right]. \tag{19}$$

The chiral anomaly[3] and the trace anomaly treated here both arise from the two basic manipulations in the path-integral formalism: We first choose the basis vectors which diagonalize the basic covariant Euclidean "energy" operator, and then examine the behavior of the functional measure defined on the basis under the (local) symmetry transformation one is interested in. If the symmetry operator and the covariant "energy" operator commute, the measure transforms trivially and no anomaly term arises. The anomaly is thus characterized by the noncommutativity of these two basic operators appearing in the theory. This can be seen in the present case by a formal introduction of the Hermitian generator W of the *global* conformal transformation $U(\alpha) \equiv e^{i\alpha W}$:

$$U(\alpha)\, g^{\mu\nu}(x)\, U(\alpha)^{-1} = e^{2\alpha} g^{\mu\nu}(x),$$
$$U(\alpha)\varphi(x)U(\alpha)^{-1} = e^{\alpha}\varphi(x) \tag{20}$$

and $U(\alpha)A_\mu(x)U(\alpha)^{-1} = A_\mu(x)$ for a generic gauge field A_μ. It is then easy to confirm that

$$i\left[W, \tfrac{1}{2}(D_\mu D^\mu - \tfrac{1}{6}R)\right] = 2\times\tfrac{1}{2}(D_\mu D^\mu - \tfrac{1}{6}R), \tag{21}$$

which may be compared with the chiral transformation

$$i\left[\gamma_5, \not{D}\right] = 2i\gamma_5\not{D}. \tag{22}$$

The relation (21) with the same weight factor 2 in the right-hand side also holds for the covariant "energy" operator of general Yang-Mills fields. For the fermion field, (21) is replaced by

$$i\left[W, \not{D}\right] = \not{D} \tag{23}$$

which arises from the tetrad $h_a{}^\mu(x)$ in $\not{D} \equiv \gamma^a h_a{}^\mu(x) \times D_\mu$; the covariant derivative D_μ here may contain the Yang-Mills (and electromagnetic) fields in addition to the gravitational field.

The anomaly factors are given by the quantum mechanical *local* expectation values of these commutators (21)–(23), as is expected from the variational derivative; for the chiral symmetry[3] one may examine an asymmetric transformation $\psi(x) \to \exp[i\alpha(x)\gamma_5]\,\psi(x)$ and $\bar\psi(x) \to \bar\psi(x)\exp[-i\alpha(x)\gamma_5]$, and for the conformal symmetry a global transformation. We thus have

$$\langle \varphi(x)[D_\mu D^\mu - \tfrac{1}{6}R]\,\varphi(x)\rangle = A_W(x) \tag{24}$$

with $A_W(x)$ in (16), and the chiral anomaly[3]

$$\langle \bar\psi(x)\, 2i\gamma_5\not{D}\psi(x)\rangle = 2\textstyle\sum_n \psi_n(x)^\dagger\gamma_5\psi_n(x) \tag{25}$$

with

$$\not{D}\psi_n(x) = \lambda_n\psi_n(x),$$
$$\int \psi_n(x)^\dagger\psi_m(x) g^{1/2} d^4x = \delta_{n,m}. \tag{26}$$

The use of the (local) index theorem[2] in (25) immediately gives rise to the well-known chiral anomaly.[1] [The zero eigenvalues of the covariant "energy" operators should be carefully treated in (24) and (25). A symmetry-breaking term such as a mass term for the fermion, for example, gives rise to an extra term proportional to the mass in (25), corresponding to the right-hand side of

$$\partial_\mu j^\mu{}_5 = 2\,{\rm Im} j_5 - \frac{i}{8\pi^2}{\rm Tr}\, {}^*F^{\mu\nu}F_{\mu\nu}$$

with $A_\mu \equiv igA_\mu{}^a T^a$.]

As was noted elsewhere,[4] the chiral anomaly does not involve any divergence and it is quite stable in the sense that any prescription of summation starting from small eigenvalues (i.e., small in their absolute values) always gives rise to an identical result in (25). In contrast, the conformal anomaly (18) diverges in the first place. It also appears to depend on the geodesic biscalar $\sigma(x,x')$ [a generalization of $\frac{1}{2}(x-x')^2$ in the flat space] if one uses the ζ regularization in the curved space,[8] and it is not easy to see whether the finite part of (18) is regularization independent. It is, however, easy to check the regularization independence of the trace anomaly associated with the background gauge field in the flat-space-time limit.[9] The relation (23) gives the (bare) trace anomaly arising from quantized fermion field in the background Yang-Mills field,

$$A_W(x) \equiv \langle \bar{\psi}(x) i \not{D} \psi(x) \rangle = \sum_n \psi_n(x)^\dagger \psi_n(x) \tag{27}$$

with the eigenfunctions (26). This may be evaluated by changing the basis to plane waves as

$$A_W(x) = \lim_{M\to\infty} \sum_n \psi_n(x)^\dagger f(\lambda_n{}^2/M^2)\psi_n(x) = \lim_{M\to\infty} \mathrm{Tr}\int \frac{d^4k}{(2\pi)^4} e^{-ikx} f((\not{D}/M)^2) e^{ikx}$$

$$= \lim_{M\to\infty} \mathrm{Tr} M^4 \int \frac{d^4k}{(2\pi)^4} f(-k^\mu k_\mu + 2ik_\mu D^\mu/M + D_\mu D^\mu/M^2 + (1/4M^2)[\gamma^\mu, \gamma^\nu] F_{\mu\nu}). \tag{28}$$

By expanding (28) in powers in $1/M$, we obtain

$$A_W(x) = \lim_{M\to\infty} \mathrm{Tr}\left\{\frac{M^4}{(4\pi)^2}\int_0^\infty dk^2\, k^2 f(k^2) - \frac{1}{24\pi^2} F^{\mu\nu}F_{\mu\nu}\right\}, \tag{29}$$

where the trace runs over the Yang-Mills internal indices. The coefficient of the finite term $F^{\mu\nu}F_{\mu\nu}$ is independent of any *smooth* regulator $f(z)$ which rapidly approaches zero at $z=\infty$ with the normalization $f(0)=1$. This shows that $A_W(x)$ has a well-defined meaning once it is renormalized at the vanishing background field, and it leads to[9]

$$\langle T_{\mu\nu}(x)\rangle_{\mathrm{ren}} g^{\mu\nu} = -\frac{1}{24\pi^2}\mathrm{Tr} F^{\mu\nu}F_{\mu\nu}$$

with $A_\mu \equiv igA_\mu{}^a T^a$.

The algebraic characterization[10] (21)–(23) defines the *anomalous symmetry,* which can potentially lead to the anomaly. The symmetry is anomalous in that it is a symmetry of the Lagrangian but not compatible with the covariant "energy" operator, which is indispensable to define the time-ordered product. The anomaly is thus best defined in the basis which diagonalizes the covariant "energy" operator; the anomalous behavior in perturbation theory is traced to the failure of a naive unitary transformation of those basis vectors to plane waves.[3,4] For a *renormalizable* theory, the *form* of the Lagrangian is not altered by higher-order effects. This property becomes important in the fully quantized theory, as my derivation of W-T identities relies on the bare form of the Lagrangian.

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