Semiclassical Approach to Planar Diagrams

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It is shown, on the basis of an SU(N)-symmetric model, that the large-N quantum theory can be described in terms of the classical equation of motion supplemented by some special boundary conditions. It is then exemplifyied how the appropriate classical solutions determine the sum of planar diagrams.

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The large-N behavior of the Yang-Mills gauge theory is given by the sum of all planar diagrams.[1-3] In the collective-field approach which was developed recently[4,5] one in general achieves the summation of large-N graphs by a single stationary point of an effective Hamiltonian. The problem is then to solve the effective field equations in explicit terms. This can be done in some simpler models but for the theory of interest, the Yang-Mills gauge theory, the problem looks rather complex.

However, in a related work of Ref. 6 it was pointed out in the framework of linear σ models that the large-N vacuum can be alternatively obtained as a special time-dependent classical solution of the original field equations. In general we have established in Ref. 7 that for any N-component vector model one can understand the effective equations as simply the classical field equations with special boundary conditions.

Here, we present the crucial generalization of this semiclassical approach to theories which have planar diagrams. As the prototype model we consider the U(N)-symmetric quantum theory of Ref. 3 with the Lagrangian

$$L=\tfrac{1}{2}\mathrm{Tr}(\dot{M}^2)-v(M), \tag{1}$$

where the basic degrees of freedom are given by the $N\times N$ Hermitian matrix $M(t)$. The interaction potential $v(M)$ is assumed to be invariant under the global SU(N) symmetry transformation

$$M\rightarrow M'=VMV^\dagger. \tag{2}$$

This implies the conservation of the "angular momentum" matrix

$$J\equiv i[M,\dot{M}]: \quad dJ/dt=0. \tag{3}$$

Let us now state in simple terms our main result. We shall show that the large-N behavior of the quantum singlet sector is directly determined by the classical equations of motion

$$d^2M(t)/dt^2+\partial v(M)/\partial M=0 \tag{4}$$

supplemented by the constraints

$$J_{ab}=\hbar(1-\delta_{ab}). \tag{5}$$

These constraints supply the nontrivial boundary conditions and signify (through $\hbar$) the quantum nature of the problem.

Before we proceed to our argument let us record the basics of the large-N effective Hamiltonian.[5] For studying the singlet subspace

$$\hat{J}_{ab}|\rangle=0 \tag{6}$$

of the quantum theory one reformulates the Hamiltonian in terms of the most general set of commuting SU(N)-invariant operators

$$\varphi(x)=\int(dk/2\pi)\exp(ikx)\,\mathrm{Tr}[\exp(-ikM)] \tag{7}$$

and the conjugate variables $\pi(x)=(h/i)\partial/\partial\varphi(x)$. This leads to the effective Hamiltonian

$$H_{\mathrm{eff}}=\int dx\left\{\frac{1}{2}\pi_{,x}\varphi\pi_{,x}+\frac{\hbar^2}{2}\varphi(x)\left[\mathrm{P}\int\frac{\varphi(y)}{x-y}\right]^2\right\}+V[\varphi], \tag{8}$$

where $V[\varphi]=v(M)$ represents the original potential while the $\hbar^2$ term

$$V_Q=\frac{\hbar^2}{2}\int dx\,\varphi(x)\left[\mathrm{P}\int\frac{\varphi(y)}{y-x}\right]^2 \tag{9}$$

is of purely quantum mechanical origin. Now, because of the constraint $\int dx\,\varphi(x)=N$ there follows the fact that the large-N limit of quantum theory is given by the stationary points of the effective Hamiltonian. The $\hbar^2$ term V_Q plays a very important role in this limit and the complexity of the effective equations comes essentially from this term.

We now begin with the consideration of the classical equations of motion. Since we shall focus our attention on the invariants let us use the "angular" decomposition[3,8]:

$$M = U\lambda U^\dagger, \tag{10}$$

where $U(t)$ is a unitary matrix with $U^\dagger U = 1$ and $\lambda(t)$ denotes the SU(N)-invariant diagonal part

$$\lambda = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_N]. \tag{11}$$

The classical equations of motion given by (4) then become

$$U\{(d/dt)(\dot\lambda + [U^\dagger \dot U, \lambda]) + [U^\dagger \dot U, \dot\lambda + [U^\dagger \dot U, \lambda]]\}U^\dagger + \partial v/\partial M = 0. \tag{12}$$

The diagonal part of this equation can be written in the form

$$\frac{d^2}{dt^2}\lambda_k(t) + \sum_{a<b} Q_{ab}Q_{ba}\frac{\partial}{\partial\lambda_k}\frac{1}{(\lambda_a-\lambda_b)^2} + \frac{\partial v}{\partial \lambda_k} = 0, \tag{13}$$

where we have defined

$$Q(t) = i\,[\lambda, [U^\dagger \dot U, \lambda]]. \tag{14}$$

This matrix represents an invariant under U(N) symmetry transformations. An alternative way to derive the above equations of motion for the invariants λ_k is to consider the Hamiltonian $H = \frac{1}{2}\mathrm{Tr}(\dot M^2) + v(M)$, and write it in the form

$$H = \frac{1}{2}\sum_{k=1}^N \dot\lambda_k^{\,2} + \sum_{a<b}\frac{Q_{ab}Q_{ba}}{(\lambda_a-\lambda_b)^2} + v(\lambda) \tag{15}$$

and show that the Poisson bracket of Q_{ab} with λ_k's vanishes.

An obvious thing about Eqs. (12) and (13) is that the angular terms do not decouple. But from our earlier experience with the large-N limit we expect to have separate equations for the invariants λ and also that all λ_k's become equally important. This could be achieved if we can introduce a constraint

$$Q_{ab}Q_{ba} = c^2(1-\delta_{ab}). \tag{16}$$

However, this quantity is *not* a constant of motion which can be easily seen since

$$Q(t) = U^\dagger(t) J U(t). \tag{17}$$

Nevertheless, we can show that the constraint (16) is consistent. Namely, if we fix the conserved quantity J as

$$J_{ab} = c(1-\delta_{ab}), \tag{18}$$

then $Q_{ab}Q_{ba}$ becomes

$$c^2[\delta_{ab} - 2(\textstyle\sum_k U_{ka}^\dagger)(\sum_{k'} U_{k'a})\delta_{ab} + |(\sum_k U_{ak}^\dagger)(\sum_{k'} U_{k'b})|^2]. \tag{19}$$

Furthermore, Eq. (18) can be written as

$$[\lambda, [U^\dagger \dot U, \lambda]]_{ab} = ic[\delta_{ab} - (\textstyle\sum_k U_{ak}^\dagger)(\sum_{k'} U_{k'b})] \tag{20}$$

which is an off-diagonal matrix and then there follows the nontrivial identity

$$(\textstyle\sum_k U_{ak}^\dagger)(\sum_{k'} U_{k'a}) = 1, \quad a = 1, 2, \dots, N. \tag{21}$$

This identity then assures that our constraint (16) can indeed be satisfied. Concerning these arguments we mention that they are similar in spirit to the ones used in demonstrating the complete integrability of the N-body Calogero system.[9-11]

Let us now show that the above classical equations with the constraints coincide in form with the effective large-N equations. For this purpose use the collective fields which according to the defining equation (7) read

$$\varphi(x) = \sum_{k=1}^N \delta(x - \lambda_k(t)), \tag{22a}$$

$$\varphi\pi_{,x} = \sum_{k=1}^N \delta(x-\lambda_k)p_k. \tag{22b}$$

One can easily show that the Poisson brackets

$$\{\varphi(x), \pi_{,y}(y)\} = \partial_y\,\delta(x-y) \tag{23}$$

are correctly satisfied. Then to perform the comparison note that the first term in the classical Hamiltonian equals

$$\tfrac{1}{2}\textstyle\sum_k p_k^{\,2} = \tfrac{1}{2}\int dx\; \varphi\pi_{,x}^{\;2} \tag{24}$$

which is the first term in the effective Hamiltonian (8). More importantly, the second, angular term in Eq. (15) can be written as

$$V_{\mathrm{ang}} = c^2\sum_{a<b}\frac{1}{(\lambda_a-\lambda_b)^2} = \frac{c^2}{2}\sum_k\left[\sum_{a\neq k}\frac{1}{(\lambda_k-\lambda_a)}\right]^2. \tag{25}$$

This rather nontrivial looking identity is easily established by the method of induction. Next, writing this in terms of the collective field $\varphi(x)$ and identifying $c^2 = \hbar^2$ we obtain the important conclusion: The angular potential (25) coincides exactly with the $\hbar^2$ quantum potential of Eq. (9):

$$V_{\rm ang} = V_Q. \tag{26}$$

We see therefore that the large-N effective Hamiltonian is nothing but a constrained classical Hamiltonian. This identification concludes our demonstration that the classical equations of motion with special boundary conditions determine directly the quantum large-N behavior.

Let us now exemplify how one can use the appropriate classical solutions to sum planar diagrams. We consider for example the potential

$$V(M) = \tfrac{1}{2}\,\mathrm{Tr}(M^2). \tag{27}$$

The classical equations now read

$$\ddot{M} + M = 0 \tag{28}$$

and the general solution is given by

$$M(t) = \lambda^0 \cos t + \Omega \sin t. \tag{29}$$

Here we have defined $\lambda^0 = \lambda(0)$ and taken $U(0) = 1$. The angular momentum constraint given by Eq. (18) determines the form of Ω:

$$\Omega_{ab} = \dot{M}(0)_{ab} = p_a{}^0 \delta_{ab} + i\hbar \frac{1-\delta_{ab}}{(\lambda_a{}^0 - \lambda_b{}^0)}. \tag{30}$$

Suppose now for example that one is interested in the large-N ground state. In the above framework it is given by the lowest-energy solution which obviously corresponds to static λ's so that $p = 0$ and λ^0 obeys the equation

$$\frac{\partial}{\partial \lambda_k{}^0}\, \frac{1}{2}\sum_a (\lambda_a{}^0)^2 + \sum_{a<b} \frac{h^2}{(\lambda_a{}^0 - \lambda_b{}^0)^2} = 0. \tag{31}$$

We now use the results of Calogero[9] which state that $\lambda_k{}^0$'s are given by the zeros of Nth-order Hermite polynomial

$$H_N(\hbar^{-1/2}\lambda_k{}^0) = 0, \quad k = 1, 2, \ldots, N. \tag{32}$$

This then completely specifies the classical solution determining the large-N vacuum.

In order to compare with earlier works let us compute for example the collective field

$$\varphi_0(x) = \sum_{k=1}^{N} \delta(x - \lambda_k{}^0). \tag{33}$$

Since we need explicitly the zeros of the Hermite polynomial we use the integral representation

$$H_N(x) = 2^{N+1}\pi^{-1/2} \exp(x^2) \int_0^\infty dt\, t^N \exp(-t^2) \times \cos(2xt - N/2\pi) \tag{34}$$

and the method of stationary phase to derive the large-N asymptotic form

$$H_N(x) \sim \cos\{N \sin^{-1}[x/(2N)^{1/2}] \times \cos[\tfrac{1}{2}x(2N - x^2)^{1/2} - \tfrac{1}{2}N\pi], \tag{35}$$

where we have assumed $x = O(N^{1/2})$. Consequently for large N (and, for example, N odd) the zeros are explicitly given by

$$\tfrac{1}{2}x_n(2N - x_n{}^2)^{1/2} = n\pi, \quad n = 0, \pm 1, \ldots, \tag{36a}$$

and by

$$N \sin^{-1}[x_m/(2N)^{1/2}] = m\pi + \pi/2, \quad m = 0, \pm 1, \ldots, \tag{36b}$$

Now one can easily show that the sum (33) (with $\hbar = 1$) converts to an integral

$$\varphi_0(x) = \pi^{-1} \int d\lambda_k (2N - \lambda_k{}^2)^{1/2}\, \delta(x - \lambda_k), \tag{37}$$

giving the result

$$\varphi_0(x) = \begin{cases} \pi^{-1}(2N - x^2)^{1/2}, & x^2 < 2N, \\ 0, & x^2 > 2N. \end{cases} \tag{38}$$

This is precisely the stationary collective field obtained in the effective-Hamiltonian approach in Ref. [5].

As the second exercise we compute directly the ground-state energy. It is given by the classical Hamiltonian (15) and for the above solution

$$E = \sum_{a<b} \frac{\hbar^2}{(\lambda_a{}^0 - \lambda_b{}^0)^2} + \frac{1}{2}\sum_k (\lambda_k{}^0)^2. \tag{39}$$

To evaluate this sum directly we employ two identities concerning the zeros of Hermite polynomials see Ref. 9. The first states that

$$\lambda_i{}^0 = \sum_{k \neq i} (\lambda_i{}^0 - \lambda_k{}^0)^{-1}$$

while the second says that the matrix

$$\delta_{ab} \sum_{k \neq a} \frac{1}{(\lambda_k{}^0 - \lambda_a{}^0)} - \frac{1 - \delta_{ab}}{(\lambda_a{}^0 - \lambda_b{}^0)^2} \tag{40}$$

has integer eigenvalues given by $n = 0, 1, 2, \ldots, N - 1$. In both statements we have scaled out $\hbar$. Using these identities we can easily show that

$$E = \sum_{n=0}^{N-1} (\hbar n) \tag{41}$$

This form is the same (apart from lower-order terms) as the quantum ground-state energy of Brézin *et al.*[3]

In conclusion let us summarize the basic facts of the semiclassical approach to large-N diagrams. On the basis of our earlier studies and the present investigation we have the following general conclusion: The large-N behavior of the quantum theory is directly determined by the classical equations of motion supplemented with some special constraints on the generators of the symmetry group. This statement is valid irrespective of whether the theory under consideration involves only bubble diagrams or the more complex planar diagrams.

The foregoing conclusion may seem surprising. Indeed there are statements in the literature that the classical solutions do not seem to be relevant in the large-N limit. Obviously, not only do we sharply disagree with such statements but moreover in the present framework we show that appropriate classical solutions in fact directly determine the large-N quantum theory.

The generalization of the above phenomenon to Yang-Mills gauge theory would be as follows. The quantum theory is given by the Hamiltonian

$$\hat{H} = \int \mathrm{Tr}[\tfrac{1}{2}\hat{E}_i^{\,2} + \tfrac{1}{4}\hat{F}_{ij}^{\,2}]\, d\vec{x} \tag{42}$$

and the requirement of gauge invariance,

$$\hat{G}(\vec{x})\,|\,\rangle = 0; \quad \hat{G}(\vec{x}) = \hat{D}\cdot\hat{E}(\vec{x}). \tag{43}$$

We then expect that the large-N behavior and the sum of all planar diagrams is given by the classical solutions of Yang-Mills equations with a special constraint $G=\rho$ representing an effective source. We have checked that this is true for the simple one-plaquette Yang-Mills theory and believe that it holds for the full three-dimensional theory.

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Measurements of Cross Sections for Pion Absorption by Nuclei

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Cross sections for true absorption by nuclei (Al, Ti, Cu, Sn, Au) of pions ($\pi^{\pm}$) in the energy range from 20 to 280 MeV were measured by a new method of detecting nuclear γ rays following the reaction. The incident-energy dependence of the cross sections in the low-energy region ($T_\pi < 50$ MeV) was well reproduced by an optical-model calculation, while the higher-energy part seems to indicate complex mechanisms of pion absorption.

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The pion absorption process is essentially an "exclusive" reaction in which no pion is left in the final state, so that its study provides unique information on pion-nucleus interactions. Despite extensive studies at meson factories, very few data have been available on the pion (true) absorption cross section.[1,2] We have measured the cross sections with a new method by detecting γ rays from residual nuclei following the pion absorption. In this Letter we shall first describe the principle and practical problems of the method, and then present experimental data.

A nucleus is excited after pion absorption and emits several nucleons until the nuclear excitation energy becomes less than the binding energy of nucleons. The nucleon emission is followed by low-energy γ-ray emissions from the excited residual nucleus with essentially the same mechanism as that well studied in the low-energy (particle, xn) reactions.[3] In the present method, we