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Quark-Quark Interaction and the Nonrelativistic Quark Model

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This Letter demonstrates that the nonrelativistic approximation breaks down for the lighter hadrons with the conventional qq one-gluon exchange potential. This is mainly due to the Coulomb and the short-range hyperfine interactions. To overcome this difficulty, some phenomenological interactions with a long-range spin dependence are proposed. The validity of treating the spin-dependent term as a perturbation is examined.

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A fundamental question pertaining to the quark model is whether it is possible to regard the hadrons as *nonrelativistic* bound systems of quarks, i.e., $q\bar{q}$ or $3q$. In order to answer this question, one may start by assuming a $q\bar{q}$ (or qq) potential, solve the two-body (or three-body) Schrödinger equation, and ascertain whether the quark velocities are indeed nonrelativistic by computing $\langle v^2/c^2\rangle$ from the kinetic energy. This has been done for charmonium,[1] where it is found that $\langle v^2/c^2\rangle \approx 0.2$ for the ground state, increasing to about 0.4 for the highly excited states. One of the aims of the present paper is to carry through this program for the ground states of the lighter baryons and the mesons. Relativistic corrections in the ground-state mass splittings of these are found to be small only with a judicious choice of the interaction, and a spin-dependent force of long range. Moreover, perturbative estimates of the spin-dependent potential are shown to be inadequate if it is of very short range.

The validity of the nonrelativistic approximation surely depends on the masses of the quarks and the form of the interaction potential chosen. Consider, for example, the hypothetical problem of a $q\bar{q}$ pair bound in a bare Coulomb potential. The energy spectrum is given by $E_n = -R/n^2$, when R is a constant and $n = 1, 2$, etc. The splitting ΔE between the $1S$ and $2P$ states is $\frac{3}{4}R$, while the kinetic energy $\langle T\rangle$ in the ground state is R. Choosing $\Delta E = 400$ MeV, and the mass m of each quark to be 336 MeV, we find $\langle T\rangle/2m = 0.8$, which is much too large. On the other hand, if the same $q\bar{q}$ pair is bound in a harmonic potential, then $\Delta E = \hbar\omega$, and $\langle T\rangle = \frac{3}{4}\hbar\omega$, so that in this case $\langle T\rangle/2m = 0.45$. This indicates that whereas a Coulomb-dominant $q\bar{q}$ potential is all right for charmonium (with $m \geq 1.6$ GeV), it is unacceptable for the lighter mesons. In this connection, we first examine the qq potential proposed by De Rújula, Georgi, and Glashow,[2] which is based on the one-gluon exchange. It is of the form

$$V_{ij} = U(r_{ij}) + k\alpha_s S_{ij}, \tag{1}$$

where U is the spin-independent universal confinement potential, and α_s is the effective gauge coupling parameter which is taken as a constant for simplicity. The constant k is $-\frac{2}{3}$ for qq and $-\frac{4}{3}$ for $q\bar{q}$. The form of S_{ij} is exactly analogous to the two-electron interaction, and when we ignore the tensor and spin-orbit parts (for S wave), contains a Coulomb term, momentum-dependent correction terms, and a hyperfine spin-dependent term proportional to $(m_i m_j)^{-1}(\vec{s}_i \cdot \vec{s}_j)$

$\times\,\delta(\vec{r}_{ij})$. Here m_i, m_j are the masses and $\vec{s}_i, \vec{s}_j$ $(=\frac{1}{2}\vec{\sigma}_j)$ the spin operators of the interacting quarks. Note that in atomic calculations hyperfine splittings are very small and may be estimated by first-order perturbation; for this reason the δ-function representation is very convenient. If, on the other hand, one wants to solve the Schrödinger equation with potential (1), $\sigma(\vec{r})$ must be replaced by an appropriate form factor $f(r, r_0)$ of range[3] r_0. Otherwise the system collapses for attractive $\delta(\vec{r})$. Since in the hadronic problem spin splittings are sizable fractions of the observed masses (e.g., between N and Δ), a perturbation treatment is questionable. We shall deal with this point later at some length.

It is important to realize that in the pioneering work of De Rújula, Georgi, and Glashow,[2] the nice fits in the baryons and 1^- mesons were obtained not by solving the Schrödinger equation, but by *parametrizing* matrix elements like $\langle\Psi|r^{-1}|\Psi\rangle$ and $\langle\Psi|\delta(\vec{r})|\Psi\rangle$. Their conclusions would not change if the radial dependences of the above two potentials were altered. The same wave function Ψ was used for all S-wave baryons, and dynamical questions, like the magnitude of $\langle v^2/c^2\rangle$, or the asymmetry in the spatial wave functions of Λ and Σ, were not investigated. Their work, therefore, had nothing to say either about the validity of the nonrelativistic approximation, or about the range of the spin-dependent force. It did show, of course, that the coefficient $(m_i m_j)^{-1}$ in the spin-dependent term played a vital role in the fit. De Rújula, Georgi, and Glashow[2] also pointed out that the potential V_{ij} that fits the S-wave baryons and the 1^- mesons cannot be expected to fit the 0^- mesons because of the importance of the two-gluon exchange term.

To our knowledge, no one has attempted to solve the three-body Schrödinger equation for the baryonic problem with the interaction given by Eq. (1). Recently, Warke and Shankar[4] attempted a variational calculation retaining the δ function in the spin-dependent term. Since the three-body Hamiltonian with an attractive δ function has no lower bound, the significance of their work is questionable. We replace it by a suitable form factor[5] $f(r, r_0)$ of range[3] r_0. If one wants to include the momentum-dependent terms in the potential (1), these must also be regularized consistently. We solve the three-body Schrödinger equation by the Feshbach-Rubinow (FR) method,[6] as generalized in other applications for unequal masses and force bonds.[7] This method assumes that the S-state three-body wave function Ψ_0 is a function of a single variable $R=\frac{1}{2}(x+y+\eta z)$, where $x=r_{23}$, and likewise for y and z, and η is a variational parameter to account for the asymmetry. The three-body Schrödinger equation then reduces to a single Schrödinger-type equation in one variable R, and may easily be solved on a computer. The minimum energy is obtained by varying η; for N and Δ, $\eta=1$ since the force bonds and the masses are identical. While we cannot give the details of the method; we emphasize that its accuracy has been satisfactorily tested both for long-range atomic problems[8] and for short-range nuclear problems[9] using central forces. Moreover, we test the accuracy of this method in the present context in an exactly solvable model, to be described later. For the interaction (1), the quark confinement potential was taken to be a ramp. For the quark masses, we put $m_u=m_d$ and m_u/m_s was restricted in the vicinity of 0.6. The parameters m_u, m_s, α_s, and an overall constant C were varied to fit the masses of $N(939)$, $\Delta(1232)$, $N^*(1470)$ and $\Lambda(1116)$. It was possible to get a number of sets. In all cases, however, the rms radius of the nucleon shrank to 0.3 fm or less, with the result that the total kinetic energy of the three quarks was >1600 MeV. The best fits were for $m_u \sim 500$ MeV, so that $\langle T\rangle/3m_u > 1$, and the system is clearly relativistic. When we recall our comments about Coulomb-dominated potentials, this result is not surprising.

In the choice of the radial forms for the interaction (1), one was guided by the ideas of quantum chromodynamics and the one-gluon exchange potential. Its essential ingredients, however, were a confinement potential and a spin-dependent term whose coefficient was mass dependent. We now show that it is possible to construct phenomenological qq interactions that retain the above two ingredients, and fit the ground-state masses with $\langle v^2/c^2\rangle < 0.5$. As an example, we propose the harmonic interaction[10]

$$V_{ij}=\vec{F}_i\cdot\vec{F}_j\{-\tfrac{1}{2}\kappa[1+(m_i m_j)^n\lambda\vec{\sigma}_i\cdot\vec{\sigma}_j]r^2+C\}, \quad (2)$$

where κ, λ, and C are adjustable constants and $\langle\vec{F}_i\cdot\vec{F}_j\rangle=-\frac{2}{3}$ for qq, $-\frac{4}{3}$ for $q\bar{q}$. The three-body problem with this interaction is exactly solvable. We find that the best fit to mass splittings is obtained when $n=-\frac{1}{2}$, and the calculated masses are displayed by the first column of numbers in Table I. We chose the quark masses $m_u=m_d$ $=336$ MeV; the constants κ, λ, and C were obtained by fitting the masses of N and Δ and the rms radius of the nucleon,[11] and $m_s=595$ MeV

TABLE I. Results of the model calculations. The kinetic energy of the three quarks in the nucleon is denoted by $\langle T\rangle_N$ and given in the first row. The rms radius of the nucleon (in femtometers) is given in the second row. All energies are in megaelectronvolts. The mass of $N(939)$ is exactly fitted in all cases. The baryon results, except in the first column, are obtained by the FR method. The interactions are adjusted to fit the baryons only. The 0^- mesons cannot be fitted by the same parameters (see text).

Model	Eq. (2) Exact	Eq. (2) FR method	Eq. (3)
$\langle T\rangle_N$	399	409	503
$\langle r^2\rangle_N^{1/2}$	0.660	0.668	0.606
$\Lambda(1116)$	1103	1106	1112
$\Sigma(1193)$	1187	1187	1194
$\Xi(1318)$	1337	1337	1336
$\Delta(1232)$	1231	1239	1234
$\Sigma^*(1385)$	1385	1391	1378
$\Xi^*(1533)$	1540	1542	1522
$\Omega(1672)$	1695	1694	1668
$N^*(1470)$	1471	1484	1476
$\rho(773)$	723		738
$K^*(892)$	893		888
$\varphi(1020)$	1060		1039

was chosen to obtain a good overall fit of the strange baryons. We found $\kappa = 437.5$ MeV fm^{-2}, $\lambda = 101.6$ MeV, and $C = 433.5$ MeV. For the three quarks in N, the total kinetic energy is 399 MeV, so that $\langle T\rangle_N/3m_u = 0.40$, corresponding to $\langle v^2/c^2\rangle = 0.44$. For Ω, the value for $\langle v^2/c^2\rangle$ is 0.32. These indicate that the $3q$ system with the interaction (2) may be regarded as marginally nonrelativistic, the $\langle v^2/c^2\rangle$ for a quark being in the same range as in the $4S$ excited state of the charmonium.[1] The nonrelativistic approximation will be questionable for the excited states of the baryons, where relativistic corrections will be much larger as compared to the ground state. We are therefore of the opinion that whereas it may be meaningful to use the nonrelativistic model for the mass splittings in the ground state of lighter baryons, spectroscopic calculations are suspect.

The interaction (2) was also used to test the FR method that we apply to solve the three-body problem. Being variational in character, it underbinds the nucleon by 19.7 MeV, and so we readjust the constant C in Eq. (2) to 443.4 MeV. The results of the FR calculation are displayed in the second column of Table I, and it is seen that the mass splittings are accurately reproduced in this approximation.

The same interaction (2) was also used to compute the masses of the 1^- S-state mesons, with $\langle \vec{F}_i\cdot\vec{F}_j\rangle = -\frac{4}{3}$. These come reasonably, as displayed in Table I. For ρ, $\langle v^2/c^2\rangle = 0.48$, and for the others it is less. We do not expect the same effective $q\bar{q}$ interaction to reproduce the 0^- mesons—as explained by De Rújula, Georgi, and Glashow.[2] We find the calculated masses of π and K to be 261 and 628 MeV, respectively, much too high.

A number of effective interactions similar to (2) may be constructed. One such potential is

$$V_{ij} = \vec{F}_i\cdot\vec{F}_j\{-\kappa[r + (m_i m_j)^{-1}\lambda\vec{\sigma}_i\cdot\vec{\sigma}_j r^{-1}] + C\}. \quad (3)$$

Using the same procedure as before, we find $\kappa = 445.6$ MeV fm^{-1}, $\lambda = 0.3625(c\hbar)^2$, $C = 631.5$ MeV, $m_u = m_d = 336$ MeV, and $m_s = 585$ MeV. The masses obtained with this potential are also displayed[12] in Table I. As suggested by Schnitzer,[13] Khare,[13] and Tabb,[13] the $1/r$-type spin dependence arises naturally from a ramp confinement if the latter is generated through a vector exchange between quarks. However, in this case $\langle v^2/c^2\rangle$ for a quark in a nucleon is 0.5, and relativistic corrections are correspondingly larger. Note that the slope of the ramp for $q\bar{q}$ system in Eq. (3) is 594 MeV fm^{-1}, which is only about two-thirds[14] the value of that in charmonium.[1]

Finally, to demonstrate the importance of the range of the spin-dependent potential and the inadequacy of the perturbation estimate, we consider the potential

$$V_{ij} = \vec{F}_i\cdot\vec{F}_j[-\tfrac{1}{2}\kappa r^2 - (m_i m_j)^{-1}\lambda f(r,r_0)\vec{\sigma}_i\cdot\vec{\sigma}_j + C]. \quad (4)$$

Here, f is the same form factor as defined before, and $m_u = m_d = 336$ MeV. The parameter κ is fixed at 241.5 MeV fm^{-2}, and, for a given choice of r_0, λ and C are determined by solving the three-body Schrödinger equation and fitting the masses of N and Δ. As the range r_0 drops much below 1 fm, we find that the kinetic energy of the quarks increases rapidly, and the system becomes relativistic.[15] For example, for $r_0 = 0.1$ fm, the kinetic energy $\langle T\rangle$ of the three quarks in the nucleon is 1711 MeV and the radius shrinks to 0.49 fm; in the perturbation approach, one uses the oscillator wave function to find that $\langle T\rangle_N$

is only 355 MeV, and the rms radius of the nucleon is 0.7 fm. Moreover, the splitting between the Δ and N, with oscillator wave functions, is only about 70 MeV, compared to 292 MeV as obtained by solving the problem dynamically. The situation improves markedly if $r_0 \geqslant 1$ fm. For example, with $r_0 = 2$ fm, the dynamical calculation yields $\langle T\rangle_N = 453.5$ MeV. In this case, the perturbation estimate for the Δ-N mass splitting is 270 MeV, quite close to the true value. We therefore think that a very short-range spin-dependent interaction would invalidate the nonrelativistic approximation, in addition to making inaccurate the perturbation estimates of the mass splittings.

In summary, we have shown that the interaction potential proposed by De Rújula, Georgi, and Glashow[2] is unsuitable for dynamical nonrelativistic calculations of light hadrons. We have also proposed some *ad hoc* effective interactions in which relativistic corrections are much smaller and which fit the ground-state masses of the lighter baryons. A long-range spin-dependent force seems to be necessary for such fits if dynamical calculations are performed.

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[1]E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D 21, 203 (1980).

[2]A. De Rújula, H. Georgi, and S. L. Glashow, Phys. Rev. D 12, 147 (1975).

[3]In the atomic case, $r_0 \approx \alpha/m_e$, where $\alpha = \frac{1}{137}$; see for example S. M. Blinder, Phys. Rev. A 18, 853 (1978). By analogy, the hadronic $r_0 \approx \alpha_s/m_q$, where m_q is a typical quark mass. This prescription would yield $r_0 < 1$ fm.

[4]R. Shankar and C. W. Warke, Tata Institute of Fundamental Research Report No. TI FR/TH/79-22, 1979 (to be published).

[5]We choose

$$f(r, r_0) = (1/2\pi r_0^{\,2})(1/r - 1/4r_0)\exp(-r/r_0),$$

which goes over to $\delta(\vec{r})$ for $r_0 \to 0$.

[6]H. Feshbach and S. I. Rubinow, Phys. Rev. 98, 188 (1955).

[7]L. Abou-Hadid and K. Higgins, Proc. Phys. Soc. 79, 34 (1962); R. K. Bhaduri, Y. Nogami, and W. Van Dijk, Nucl. Phys. B1, 269 (1967), and B2, 316(E) (1967).

[8]R. K. Bhaduri and Y. Nogami, Phys. Rev. A 13, 1986 (1976).

[9]M. McMillan, Can. J. Phys. 43, 463 (1965).

[10]An alternative, and even simpler form of harmonic interaction is

$$V_{ij} = \vec{F}_i \cdot \vec{F}_j\,[-\tfrac{1}{2}\kappa r^2 - (m_i m_j)^{-1}\lambda\,\vec{\sigma}_i \cdot \vec{\sigma}_j + C].$$

A fit comparable to that displayed in Table I is obtained by taking $m_u = m_d = 336$ MeV, $m_s = 574$ MeV, $\kappa = 304.2$ MeV fm^{-2}, and $\lambda/m_u^{\,2} = 73.2$ MeV. This interaction for the nonstrange baryons was suggested by D. A. Liberman, Phys. Rev. D 5, 1542 (1977).

[11]We fit the rms radius in the range 0.6 to 0.7 fm, which is somewhat smaller than the observed charge radius of the proton. Our choice is also guided by the fit to N^* mass, although this is questionable in view of the larger relativistic correction in the excited state.

[12]The variational parameter η in the FR method is different from unity when the three quark masses, or the force bonds, are unequal. With the interaction (3), for example, η is 1.04 for Λ and is 0.68 for Σ, showing marked asymmetry.

[13]H. J. Schnitzer, Phys. Lett. 65B, 239 (1976); A. Khare, Phys. Rev. D 18, 4282 (1978); C. H. Tabb, Phys. Rev. D 20, 1746 (1979).

[14]The energy spectrum of a $q\bar{q}$ system in a ramp of slope K is determined by the parameter $(K^2/m)^{1/3}$, where m is the mass of a quark [J. F. Gunion and R. S. Willey, Phys. Rev. D 12, 174 (1975)]. This parameter is about 20–25% larger in the lighter mesons than in charmonium, reflecting somewhat larger energy gaps between the excited states. This implies that the effective interactions that we are proposing for the lighter hadrons would not fit the charmonium *spectrum*, although the ground-state mass may still be fitted with a reasonable choice of the charmed-quark mass m_c. For example, with the given parameters of potential (2), $m_c \approx 1.7$ GeV yields the mass of $\psi(3095)$.

[15]This point is also verified in the $q\bar{q}$ system. For example, let us consider the K meson, and take $m_s = 500$ MeV. Keeping the parameters κ and C in (4) the same as before, the Schrödinger equation is solved and λ is adjusted for a choice of r_0 to fit the K mass. We find that for $r_0 = 5$ fm, the kinetic energy of the $q\bar{q}$ system is 300 MeV, but it rises to about 900 MeV when $r_0 = 0.2$ fm.