Periodic Spontaneous Collapse and Revival in a Simple Quantum Model

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(Received 7 December 1979)

This Letter reports on the existence of periodic spontaneous collapse and revival of coherence in the dynamics of a simple quantum model. Also given are the first accurate expressions for the intermediate-time and long-time dynamical behavior of the model.

PACS numbers: 32.80.Bx, 42.50.+q

The coherent-state Jaynes-Cummings model[1] (a relative of the Lee model of quantum field theory[2] and essentially identical to the spin-phonon model of NMR) has been studied many times because of the relatively realistic way that it represents the quantum physics of a resonant interaction. It is the simplest fully quantized model of interest in NMR, quantum optics, quantum electronics, and resonance physics in general. From this model one hopes to learn, for example, about the role of quantum mechanics in the coherence properties of interesting radiation-matter or spin-lattice systems.

Despite the importance of this model, its dynamics have never been fully investigated because of the central role played by the infinite sum:

$$\langle \hat{\sigma}_z(t)\rangle = -\exp[-|\alpha|^2]\sum_0^\infty (n!)^{-1}|\alpha|^{2n} \times \cos 2\lambda(\bar{n})^{1/2}t. \qquad (1)$$

This sum has no known finite analytic expression. The physical significance of the sum is that it represents on a scale between -1 and $+1$ the degree of excitation of a two-level system. The excitation is brought about by a quantized Hamiltonian interaction between an atom (or spin) and a single mode of a quantized radiation field (or phonon bath). The quantum statistical state of the field is, at $t=0$, the fully coherent state $|\alpha\rangle$, and λ is the atom-field coupling constant.

The Hamiltonian for the model[3] can be expressed in terms of Pauli matrices and creation and destruction operators for the field mode:

$$\hat{H} = \hbar\omega_0(\hat{\sigma}_+\hat{\sigma}_- - \tfrac{1}{2}) + \hbar\omega\hat{a}^\dagger a + \hbar\lambda(\hat{a}^\dagger\hat{\sigma}_- + \hat{\sigma}_+\hat{a}). \qquad (2)$$

The commutators of the model, namely $[\hat{a}, \hat{a}^\dagger]=1$, $[\hat{\sigma}_+, \hat{\sigma}_-]=\hat{\sigma}_z$, etc., are completely conventional. Note the mixture of Bose and Fermi (harmonic oscillator and angular momentum) dynamical variables in the Hamiltonian. Even though there are only two physical degrees of freedom in the problem, a finite-normal-mode analysis of the dynamics has never been carried out.

The coherence properties of (2), for short times ($\lambda t \lesssim 10$), were first considered by Cummings[4] in 1965, and by many other authors since. While the Hermitean character of the Hamiltonian guarantees loss-free evolution for all dynamical variables, it is also known from the work of Cummings,[4] Stenholm,[5] Meystre *et al.*,[6] and Von Foerster[7] that the model exhibits decay and the onset of randomness.

In this paper we report what are apparently the first results, either analytic or numeric, for the model that are accurate for nearly arbitrary times ($0 \leq \lambda t \leq 10^6$). We have obtained long-time analytic approximations for various dynamical variables in the model, and accurate numerical representations of them. We have found (a) an analytic expression, and new interpretation, for a short-time "collapse function"; (b) the numerical observation, and analytic confirmation, of the existence of long-time "revivals" or quasi-recorrelations of initial coherence; (c) the numerical observation, and analytic confirmation, that these revivals recur periodically; and (d) the analytic prediction, and numerical confirmation, that the revivals are bounded by a monotonically decreasing envelope.

It should be emphasized at the outset that we believe these revivals are not associated with Poincaré-type recurrence. We can discuss briefly the derivation of these results, and display some of them.

For very short times and very large $\bar{n} = |\alpha|^2$ (which we will refer to as the photon number), the sum in (1) behaves like $\cos[2\lambda(\bar{n})^{1/2}t]$. Cummings first showed[4] that, when $\omega=\omega_0$ and for intermediate values of t, the cosine oscillations "collapse". They are terminated by the Gaussian envelope $\exp[-\frac{1}{2}(\lambda t)^2]$. For the same intermediate range of times we have found an improvement on Cummings's collapse function. It is valid for arbitrary $\Delta \equiv \omega_0 - \omega$ and for intermediate as well as very large values of $\bar{n}$:

$$\exp[-\tfrac{1}{2}(\lambda t)^2] \to \exp\left(-\frac{2\bar{n}\lambda^2}{\Delta^2+4\bar{n}\lambda^2}\right)(\lambda t)^2. \qquad (3)$$

Our collapse function has an $\bar{n}$ dependence that enters nonlinearly with the coupling constant λ. However, a natural interpretation is given below. The new collapse function (3) is in excellent agreement with new numerical evaluations. This is shown in Fig. 1 for $\bar{n}=25$ photons. The significantly slower collapse off resonance ($\Delta \neq 0$) predicted by (3) is apparent.

For longer times, the extremely rapid temporal oscillations of $\cos[2\lambda(\bar{n})^{1/2}t]$ for $\lambda^2\bar{n} \gtrsim 100$ have made numerical studies of the model occasionally unreliable in the past. By long times we mean $2\lambda(\bar{n})^{1/2}t \gtrsim 10^2-10^3$, i.e., long after the collapse of coherence is complete. There have been no analytic studies at all in this time domain. However, by repeated interaction between analytic and numeric approaches to the problem, we have developed reliable techniques[8] for the study of times as long as $2\lambda(\bar{n})^{1/2}t \sim 10^6-10^8$.

The analytic expression for $\langle\hat{\sigma}_z(t)\rangle$ that we have derived is

$$\langle\hat{\sigma}_z(t)\rangle \simeq \frac{\Delta^2}{\Omega^2}+\frac{4\lambda^2\bar{n}}{\Omega^2}\left(1+16\frac{\lambda^8\bar{n}^2t^2}{\Omega^6}\right)^{-1/4} e^{-\varphi(t)}\cos\Phi(t). \qquad (4)$$

Here the two phase functions have the following forms:

$$\varphi(t) = 2\bar{n}\sin^2(\lambda^2t/\Omega)(1+16\lambda^8\bar{n}^2t^2/\Omega^6)^{-1},$$

$$\Phi(t) = \Omega t+\bar{n}\sin(2\lambda^2t/\Omega) - 2\lambda^2\bar{n}t/\Omega - \tfrac{1}{2}\tan^{-1}(4\lambda^4\bar{n}t/\Omega^3),$$

where t cannot be so large that neighboring revivals overlap strongly, and where $\Omega^2=\Delta^2+4\lambda^2\bar{n}$ can be recognized as the square of the Rabi frequency for the corresponding semiclassical problem.[9]

The most striking feature of the long-time regime is that the Gaussian collapse implied by (3) is quasireversible. We have found that the model contains a "revival time" T_R whose analytic expression is

$$T_R = \pi\lambda^{-2}(\Delta^2+4\lambda^2\bar{n})^{1/2}. \qquad (5)$$

At all times $t=kT_R$, $k=1, 2, 3, \ldots,$ the quantity $\langle\sigma_z(t)\rangle$ spontaneously revives from nearly zero to a value ≈ 1 and then redecays. These revivals or quasirecorrelations and their period were first found numerically, and then confirmed analytically. Our analysis provides the revival-time (5) collapse function (3), and is most accurate in the vicinity of the revivals, but it also gives a bound $B(t)$ on the peak height achieved in the revivals:

$$B(t) = [1+16\lambda^8\bar{n}^2t^2/(\Delta^2+4\lambda^2\bar{n})^3]^{-1/4}. \qquad (6)$$

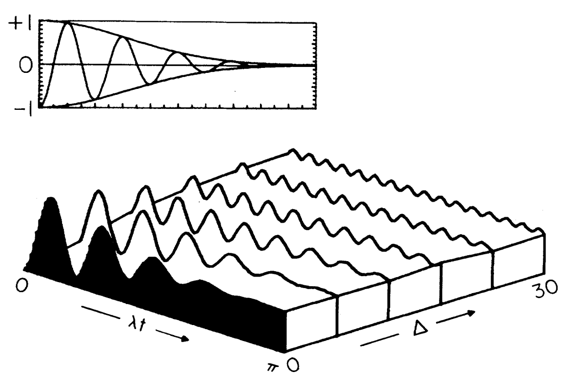

FIG. 1. Plots of the short-time behavior of $\langle\hat{\sigma}_z(t)\rangle$ for $|\alpha|^2=\bar{n}=25$ and for six evenly spaced values of Δ. The lack of reference to either absolute frequency ω or ω_0 is a feature of a rotating-wave theory (see Ref. 9) and $|\Delta| \ll \omega, \omega_0$ is implied. In the small box the $\Delta=0$ curve is drawn for the same time scale, and the short-time collapse function given in (3) is included.

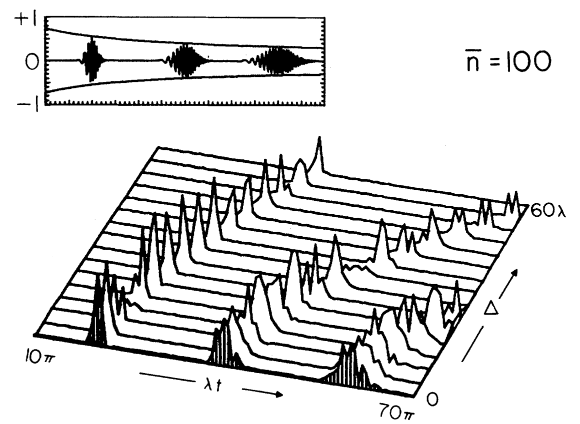

FIG. 2. A map of the t-Δ plane showing the earliest revival points of the solution for $\langle\hat{\sigma}_z(t)\rangle$. A numerical enhancement has been used to emphasize the revivals on the map, and the specific shapes of individual revivals are not to be taken seriously. In the small box the first three revivals for $\Delta=0$ are plotted without enhancement and with much greater resolution over the same time interval used for the map. The excellent fit of the long-time envelope function $B(t)$ is evident.

This envelope is also in good agreement with new numerical results, as is shown in Fig. 2.

The analytic-numeric results presented here show new features of the loss-free Hamiltonian dynamics of resonantly interacting quantum systems. Any dependence on the nonvanishing of $[\hat{a}, \hat{a}^\dagger]$ can be said to be a pure quantum effect. There are three such effects that stand out in our results. First, the discrete interval in (1) that separates the nth and the $(n+1)$th terms can be shown to derive directly from $[\hat{a}, \hat{a}^\dagger]=1$. This discreteness, in turn, implies that, after a certain interval of time, neighboring terms in (1) recover their original phase relation, and the interval is precisely the revival time T_R given in (5). Second, our interpretation of the generalized envelope (3) suggests a role for quantum effects in collapse as well as in revival. Apart from the factor $\lambda^2 t^2$, the exponent in (3) is the probability of occupying the upper state $|+\rangle$ (where $\hat{\sigma}_z|+\rangle = |+\rangle$) in the fully power-broadened steady-state limit of excitation by a strong nonquantized field whose intensity corresponds to $\bar{n}$ photons or phonons. That is, collapse occurs only if the upper state is occupied (i.e., only if spontaneous emission can occur). Third, the quantum stochasticity of the interaction, originating at least partially in the nonvanishing of $[\hat{a}, \hat{a}^\dagger]$, eventually leads to randomness in the time record of every expectation value. This "irreversibility" sets in very early if $\bar{n}$ is small enough, as Fig. 3 shows, and is expected to occur well in advance of any Poincaré recurrences. This is in agreement with preliminary estimates of Drummond and Yeh.[10]

Our work also shows clearly the origin of much misunderstanding of earlier numerical work. Our formulas (3) and (5) make it clear that values of $\bar{n}$ smaller than 10–20 are not large enough to separate the tail of the beginning of the first revival. Only with the computation and display of results for $\bar{n} \gtrsim 10^2$ and $\lambda t \gtrsim 10^2$ do the collapse

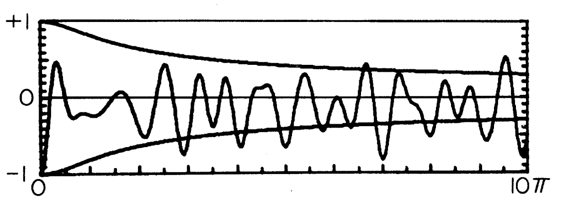

FIG. 3. Plot of the sum given in (1) for $\alpha = \sqrt{2}$ as a function of λt. The smoothly decreasing lines represent the envelope function $B(t)$ defined in (6). It provides a surprisingly good bound even here where $\alpha \sim 1$ instead of $\gg 1$.

and revival of the model's dynamic quantum coherence become recognizable.

A much more detailed description of the coherence properties of the model, with use of the same techniques used in obtaining the results described here,[8] can be obtained from a study of the multiple-time correlation functions[11] in the theory. The second-order correlations allow a discussion of the system's emission spectrum and of the photon statistics. These further results, and the role of collapse and revival in them, as well as a full description of the analytic and numeric methods we have used, will be reported elsewhere.[12]

We are pleased to acknowledge conversations with P. D. Drummond, J. R. Klauder, P. Meystre, E. W. Montroll, S. Stenholm, D. F. Walls, and J. J. Yeh, and partial financial support from the U. S. Department of Energy. H. I. Yoo assisted with the final formulation of the phases $\varphi(t)$ and $\Phi(t)$. One of us (N.B.N.) was an International Research and Exchange Board Visiting Scholar during 1978-1979. One of us (J.J.S.M.) is a recipient of a Mary Street Jenkins Foundation Fellowship.

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[1]E. T. Jaynes and F. W. Cummings, Proc. Inst. Elect. Eng. 51, 89 (1963). See also E. T. Jaynes, Stanford University Microwave Laboratory Report No. 502, 1958 (unpublished); F. W. Cummings, Ph.D. Thesis, Stanford University, 1963 (unpublished).

[2]See, for example, S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson, & Co., Evanston, 1961), p. 352.

[3]Counterrotating-wave terms, e.g., $\hat{\sigma}_-\hat{a}$ and $\hat{a}^\dagger\sigma_+$, are (following convention for this model) not included in $\hat{H}$. Their exclusion prevents the model from considering Bloch-Siegert–type effects.

[4]F. W. Cummings, Phys. Rev. 140, A1051 (1965).

[5]See the review by S. Stenholm, Phys. Rep. 6C, 1 (1973).

[6]The works by P. Meystre, A. Quattropani, and coworkers are summarized in P. Meystre *et al*. Nuovo Cimento 25, 521 (1975).

[7]T. von Foerster, J. Phys. A 8, 95 (1975).

[8]Our analytic solution procedure is long but not complicated. It is based on the assumption $|\alpha| \gg 1$ (but it gives qualitatively excellent results even for $|\alpha| = 2$). Exact infinite-series solutions for expectations of the model's dynamical variables (for several examples, see Ref. 5) are approximated by Euler-MacClaurin integrals, which are in turn approximately evaluated with saddle-point methods. The details will be discussed elsewhere.

[9]See, for example, L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New

York, 1975), Sect. 7.3.
[10]P. D. Drummond and J. J. Yeh, private communication.
[11]These may be found analytically with known Heisenberg-picture operator solutions: J. R. Ackerhalt, Ph.D. thesis, University of Rochester, 1974 (unpublished). See also J. R. Ackerhalt and K. Rzażewski, Phys. Rev. A 12, 2549 (1975), Sect. IV D.
[12]J. J. Sanchez-Mondragon, N. B. Narozhny, and J. H. Eberly, to be published.

Vacancy-Carbon Interaction in Iron

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(Received 24 January 1980)

Electron-irradiated high-purity α-iron doped with various amounts of interstitial carbon impurities has been studied by positron-lifetime measurements. It is shown that during vacancy migration at 220 K an asymmetric vacancy-carbon pair is formed, where the carbon atom is located off the center of the vacancy.

PACS numbers: 61.70.Bv, 61.80.Fe, 78.70.Bj

The properties of atomic defects as well as their interactions, especially in bcc metals, are not well known in spite of their great technological importance.[1,2] The most important example in this respect is carbon in iron, where the characteristics of vacancies and interstitials have been difficult to extract. Recent experiments,[3] utilizing positron-annihilation technique,[4] revealed that vacancy migration occurs already around 220 K. At this temperature the interstitial carbon atoms are still immobile.[5] A contradictory view of vacancy migration, based mainly on the vacancy migration energy derived from high-temperature data, has also been presented.[6]

In this Letter we report positron-lifetime measurements of the interaction of monovancies and interstitial carbon impurities in electron-irradiated α-iron. Positrons are sensitive to vacancy-type defects, whereas interstitials and their agglomerates do not affect the annihilation characteristics. Positron localization at defect sites gives information on both the concentration and the internal structure of the defects.[4] Our results show that the migration of vacancies at 220 K leads to the formation of asymmetric carbon-vacancy pairs.

The high-purity α-iron samples were prepared by zone-refining methods described earlier.[3] The interstitial impurity concentration (carbon and nitrogen) was below 5 ppm. Carbon doping was made in the liquid phase followed by solution annealing at 750 °C. The specimens were then quenched into icy water, electrolytically polished, and stored at liquid nitrogen temperature. Two sets of samples were prepared with carbon concentrations of 50 and 750 ppm. Two identical samples ($6\times8\times0.3$ mm^3) from both sets were electron irradiated at 20 K with 3-MeV electrons to a total dose of about 3×10^{19} e^-/cm^2. Positron-lifetime measurements (resolution 290 psec full width at half maximum were performed after isochronal (30 min) annealing treatment of the samples. The lifetime spectra were measured at 77 K up to 340-K annealing, whereafter measurements were made at room temperature. After source-background corrections, the lifetime spectra were analyzed with use of one or two exponential components.

Figure 1 gives the shorter positron lifetime values τ_1 as a function of the isochronal annealing temperature for both carbon-doped, electron-irradiated Fe specimens. For comparison, we also give the τ_1 values for an undoped (< 5 ppm C) Fe sample with an irradiation dose of only 6×10^{18} e^-/cm^2 (refered to as the high-dose sample in Ref. 3). The longer lifetime values τ_2 as well as the relative intensities I_2 of the longer component are shown in Fig. 2. The corresponding data for pure iron are given in Ref. 3.

At annealing temperatures below 200 K all life-

