Colored Monopoles

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A complete set of the pointlike monopole solutions for the group SU(3) is presented which can describe arbitrary allowed magnetic charges. The method adopted to construct the solutions makes use of the topological structure of the non-Abelian symmetry and could be applied to an arbitrary gauge group G.

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Non-Abelian gauge theory has become well known to have nontrivial topological structure[1] intrinsic to the symmetry group. The topological structure could be exhibited as classical solutions[2] of the system which carry nonvanishing magnetic charges conserved for topological reasons. In the meantime the possible physical importance of the topological structure and the corresponding classical solutions of non-Abelian theory in connection with the color confinement in quantum chromodynamics (QCD) have been emphasized by many authors.[3,4] Indeed, based on the group SU(2), it has recently been argued that the topological structure and the monopole solutions expressed in the form of the *magnetic symmetry* play a crucial role to establish the duality[5,6] which exists in non-Abelian gauge theory and to provide us with the magnetically condensed vacuum necessary for the confinement of color in QCD. Thus it is desirable to obtain all the monopole solutions for the color gauge group SU(3) and to clarify how these solutions exhibit the topological structure of the group. Although a large volume of the literature already exists on this issue,[7] a complete set of solutions for arbitrary allowed color magnetic charges has so far not been obtained. The purpose of this Letter is to construct such a complete set of the monopole solutions for QCD and to clarify their topological meaning.

Since it is unlikely for one to obtain the desired solutions just by studying and solving the equations of motion, I will choose a completely different tactic to attack the problem. Instead of *solving* the equations of motion, I will first examine the topological structure of non-Abelian gauge symmetry in detail and show how one can *construct* the solutions directly by making use of its topological properties. To achieve the goal it is crucial to understand that non-Abelian gauge symmetry allows an additional internal symmetry called the magnetic symmetry,[5,6] which can manifest the topological structure of the symmetry group. The magnetic symmetry has been introduced as a set of self-consistent Killing vector fields of the internal (i.e., the group) fiber space which restricts some of the dynamical degrees of freedom while keeping the full gauge symmetry intact. One of the virtues of the magnetic symmetry for our purpose is the fact that it is best suited to describe the topological structure and thus the monopole solutions of the underlying symmetry group. Although this point has been demonstrated before[5] with the group SU(2), I will argue in this Letter that this is true for an arbitrary group G by showing that a proper magnetic symmetry applied to SU(3) can provide us with a complete set of the monopole solutions.

I will start by briefly reviewing the magnetic symmetry.[5,6] Consider the higher-dimensional unified metric formulation[8,9] of the gauge theory and let ξ_i ($i=1,2,\ldots,n$) be a set of internal Killing vector fields that satisfies the canonical commutation relations of the isometry group G,

$$[\xi_i,\ \xi_j]=f_{ij}{}^k\xi_k. \tag{1}$$

The existence of these Killing vector fields at each space-time point guarantees us the gauge symmetry. A magnetic Killing vector field m is then defined as an *additional* Killing vector field which is internal,

$$m=m^i\xi_i, \tag{2}$$

and which commutes with ξ_i,

$$[m,\ \xi_i]=0\quad(i=1,2,\ldots,n). \tag{3}$$

From (2) and (3) it follows that the multiplet $\hat{m}$ defined by

$$\hat{m}=\begin{pmatrix}m^1\\ m^2\\ \cdot\\ \cdot\\ \cdot\\ m^n\end{pmatrix}$$

forms an adjoint representation of the group. In

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terms of $\hat{m}$ the Killing symmetry assumption[5,6] can then be written as

$$D_\mu \hat{m} = \partial_\mu \hat{m} + g\vec{B}_\mu \times \hat{m} = 0, \tag{4}$$

where $\vec{B}_\mu$ is the gauge potential of the group G, when the internal metric is the Cartan-Killing form. This implies that one can always normalize $\hat{m}$ to satisfy $\hat{m}^2 = 1$, which we will do in the following.

To show the importance of the magnetic symmetry for our purpose let us briefly review how[5] a proper choice of the magnetic symmetry can provide us with a complete set of the monopole solutions for SU(2). For the group SU(2) there can be only one (nontrivial) magnetic Killing vector $\hat{m}$. Because of the assumption (4) the gauge potential must have the form

$$\vec{B}_\mu = A_\mu \hat{m} - g^{-1}\hat{m} \times \partial_\mu \hat{m}, \tag{5}$$

where A_μ is an Abelian potential which is not fixed by the magnetic symmetry (4). Now let us choose

$$\hat{m} = \exp(-n\varphi t_3)\exp(-\theta t_2)\hat{\xi}_3 = \begin{pmatrix} \sin\theta\cos(n\varphi) \\ \sin\theta\sin(n\varphi) \\ \cos\theta \end{pmatrix}, \tag{6}$$

where n is an integer, θ and φ are the angular spherical coordinates of S_R^2, the two-dimensional sphere of the three-dimensional space, and t_i ($i = 1, 2, 3$) are the adjoint representation of the SU(2) generators. Clearly the above $\hat{m}$ describes all the possible homotopy class of the mapping $\pi_2(S^2)$ such that

$$\hat{m}:\ S_R^2 \to S^2 = \mathrm{SU}(2)/\mathrm{U}(1), \tag{7}$$

with the homotopy class $Z = n$. Now it is a trivial matter to confirm that the potential (5) with $A_\mu = 0$ and $\hat{m}$ given by (6) describes a complete set of the pointlike monopole solutions for SU(2). In fact, the corresponding field strength $\vec{G}_{\mu\nu}$ is given by

$$\vec{G}_{\mu\nu} = -(n/g)\sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\nu\theta\partial_\mu\varphi)\hat{m}, \tag{8}$$

which describes the monopoles with the magnetic charges $g_m = 4\pi n/g$. The topological meaning of the solutions in connection with the magnetic symmetry m has been emphasized before.[5]

From the above example it becomes clear that one may be able to find a complete set of the monopole solutions for an arbitrary group, in particular for SU(3), by imposing a proper magnetic symmetry to the potential. For SU(3) the relevant homotopy group is $\pi_2[\mathrm{SU}(3)/\mathrm{U}(1)\otimes\mathrm{U}'(1)]$ so that

$$\pi_2[\mathrm{SU}(3)/\mathrm{U}(1)\otimes\mathrm{U}'(1)] = \pi_1[\mathrm{U}(1)\otimes\mathrm{U}'(1)], \tag{9}$$

where $\mathrm{U}(1)\otimes\mathrm{U}'(1)$ is the two Abelian subgroups generated by λ_3 and λ_8, so that the monopoles must now be classified by two integers. The generalized quantization condition can be written as[10]

$$\exp[4\pi i g(\tfrac{1}{2}\lambda_3 g_m + \tfrac{1}{2}\lambda_8 g_m')] = 1$$

or

$$g_m = g^{-1}(n - \tfrac{1}{2}n'), \quad g_m' = \tfrac{1}{2}\sqrt{3}\,(g^{-1}n'), \tag{10}$$

where n and n' are integers. Then, our task here is to construct one monopole solution for every set of integers n and n' by finding a proper magnetic symmetry which could manifest the mapping,[9] and then by imposing the symmetry to the gauge potential. For this purpose we first observe that *if $\hat{m}_1$ and $\hat{m}_2$ are two Killing vectors* the self-consistency requires that—not only their antisymmetric product (i.e., the f product) $\hat{m}_1 \times \hat{m}_2$ but also—*their symmetric product (i.e., the d product) $\hat{m}_1 * \hat{m}_2$ has to be another Killing vector.* Here we denote the symmetric product by $*$. This observation follows from the following simple identities

$$D_\mu(\hat{m}_1 \times \hat{m}_2) = (D_\mu\hat{m}_1)\times\hat{m}_2 + \hat{m}_1 \times (D_\mu\hat{m}_2), \quad D_\mu(\hat{m}_1 * \hat{m}_2) = (D_\mu\hat{m}_1) * \hat{m}_2 + \hat{m}_1 * (D_\mu\hat{m}_2). \tag{11}$$

From this it becomes clear that a magnetic Killing vector $\hat{m}$ *automatically* generates another one $\hat{m}'$,

$$\hat{m}' = \sqrt{3}\,\hat{m} * \hat{m}, \tag{12}$$

unless $\hat{m}'$ is equal to $\hat{m}$. Obviously for a complete set of the monopole solutions two Killing vectors $\hat{m}$ and $\hat{m}'$ are necessary and sufficient. Furthermore, one can show that *one may choose the fundamental symmetry m to be always λ_3-like, in which case the product symmetry $\hat{m}'$ automatically becomes λ_8-like.* The potential that satisfies the condition (4) should then have the following form

$$\vec{B}_\mu = A_\mu\hat{m} + A_\mu'\hat{m}' - g^{-1}\hat{m}\times\partial_\mu\hat{m} - g^{-1}\hat{m}'\times\partial_\mu\hat{m}', \tag{13}$$

where A_μ and A_μ' are the components which are not fixed by the condition (4). Now we have to choose $\hat{m}$ in such a way as to exhibit the full homotopy class of the mapping (9). Let $\hat{m}$ be

$$\hat{m}=\exp[-n'\varphi(-\tfrac{1}{2}t_3+\tfrac{1}{2}\sqrt{3}\,t_8)]\times e^{-\theta t_n}\exp[-(n-\tfrac{1}{2}n')\varphi t_3 e^{-\theta t_2}]\hat{\xi}_3, \quad (14)$$

where t_i $(i=1,2,\ldots,8)$ are the adjoint representations of the SU(3) generators. Clearly the $\hat{m}$ exhibits the homotopy class which consists of n windings of the i-spin subgroup followed by n' windings of the u-spin subgroup of SU(3), and thus makes an obvious candidate to represent the mapping.[9] Explicitly, we find

$$\hat{m}=\begin{pmatrix} \sin\theta\cos\tfrac{1}{2}\theta\cos[(n-n')\varphi] \\ \sin\theta\cos\tfrac{1}{2}\theta\sin[(n-n')\varphi] \\ \tfrac{1}{4}\cos\theta(3+\cos\theta) \\ \sin\theta\sin\tfrac{1}{2}\theta\cos(n\varphi) \\ \sin\theta\sin\tfrac{1}{2}\theta\sin(n\varphi) \\ -\tfrac{1}{2}\sin\theta\cos\theta\cos(n'\varphi) \\ -\tfrac{1}{2}\sin\theta\cos\theta\sin(n'\varphi) \\ \tfrac{1}{4}\sqrt{3}\cos\theta(1-\cos\theta) \end{pmatrix}. \quad (15)$$

As the last step we still have to make sure that with the above $\hat{m}$, and with proper choices of A_μ and A_μ' the potential (13) does describe the desired solutions. This can be achieved by choosing

$$A_\mu=-\tfrac{1}{2}(n'/g)\sin^2\theta\,\partial_\mu\varphi,\; A_\mu'=0. \quad (16)$$

Indeed, after a straightforward calculation one can show that the potential (13) with (15) and (16) can be gauge transformed to

$$\vec{\mathrm{B}}_\mu \to g^{-1}[(n-\tfrac{1}{2}n')\hat{\xi}_3+\tfrac{1}{2}\sqrt{3}\,n'\hat{\xi}_8]\cos\theta\,\partial_\mu\varphi \quad (17)$$

[up to the residual $U(1)\otimes U'(1)$ gauge degrees of freedom] in the magnetic gauge where $\hat{m}$ and $\hat{m}'$ become space-time–independent $\hat{\xi}_3$ and $\hat{\xi}_8$. Although in the magnetic gauge the potential (17) appears to have the string singularity along the z direction *the smoothness of the potential (13) in the original gauge is guaranteed by the smoothness of A_μ and $\hat{m}$ everywhere except at the origin.* The singular potential (17) implies that in the original gauge one must have

$$\vec{\mathrm{G}}_{\mu\nu}=-g^{-1}[(n-\tfrac{1}{2}n')\hat{m}+\tfrac{1}{2}\sqrt{3}\,n'\hat{m}']\times\sin\theta(\partial_\mu\theta\partial_\nu\varphi-\partial_\nu\theta\partial_\mu\varphi), \quad (18)$$

which can easily be confirmed by a straightforward calculation. The magnetic charges g_m and g_m' of the solutions can then be defined as

$$g_m=\int_{S_R{}^2}\hat{m}\cdot\vec{\mathrm{G}}_{\mu\nu}\,d\sigma^{\mu\nu}=(4\pi/g)(n-\tfrac{1}{2}n'),$$
$$g_m'=\int_{S_R{}^2}\hat{m}'\cdot\vec{\mathrm{G}}_{\mu\nu}\,d\sigma^{\mu\nu}=\tfrac{1}{2}\sqrt{3}\,(4\pi/g)n'. \quad (19)$$

Clearly in the original gauge the potential (13) with (15) and (16) will satisfy the classical equations of motion everywhere *except* at the origin, where the Bianchi identity is violated because of the presence of the pointlike magnetic charges. In conclusion, the potential (13) with (15) and (16) indeed describes all the homotopically inequivalent pointlike colored monopoles. It is amusing to see that the topological considerations alone (with a little bit of guess work without much use of the equations of motion) allow one to construct a complete set of the monopole solutions.

After we have achieved our goal to construct all the monopole solutions for SU(3), a few comments are in order. First it has often been claimed that to define a proper homotopy (and thus a proper topological charge) for non-Abelian gauge theory one needs to have a scalar multiplet explicitly in one's theory as in the Higgs-type theory. The present analysis shows that this is simply not true. After all, one knows that the homotopy of the symmetry is determined by the group itself with no reference to any Higgs field. Of course, Higgs field becomes necessary[2,11] if one wants to make the above solutions smooth everywhere *including* the origin. A systematic analysis on the colored monopoles with Higgs field will be presented in a separate paper.[12] Secondly, it has often been claimed that some of the known SU(3) monopole solutions[7] have a λ_8-like symmetry. Again the present analysis shows that this is misleading. The true symmetry of the solutions must *always* be λ_3-like, which automatically generates a λ_8-like symmetry. Finally, we should like to emphasize *the generality of the present method* to construct directly a complete set of the monopole solutions. The method is in a drastic contrast to the conventional way to find the solutions by solving the equations of motion with a reasonable *Ansatz*. Although I have successfully applied the method to SU(2) and SU(3), it becomes clear that the method could, in principle, be applied to an arbitrary group. Furthermore, this method provides one with a deeper insight to the topological structure of non-Abelian gauge symmetry, telling one how the topological structure exhibits itself in the form of the monopole solutions.

The method presented above to construct the monopole solutions deserves to be noted in its own right. But perhaps a more important physical implication of it is that it may be used to establish the duality that exists in QCD, and thus allows one to obtain the monopole condensation for the QCD vacuum. This point could be demonstrated in the same way that it has been done before[5,6] with the group SU(2). This issue will be discussed in detail in a forthcoming communication.

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Do Total Cross Sections for Scattering of Off-Shell Particles Grow Like a Power of the Energy?

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Let $\sigma_P(Q^2, s)$ be the total cross section for scattering of meson P on some target, with $-\boldsymbol{Q}^2$ the off-shell mass of P. It is shown that, when $-Q^2 \neq m_P^2$ (the physical $\boldsymbol{P}$ mass), quantum chromodynamics suggests that, as the c.m. energy squared s approaches ∞,

$$\sigma_P(Q^2, s) \sim C_1 f(Q^2) s^\lambda + \ldots, \quad \lambda > 0,$$

where $f(Q^2)$ is calculable for large Q^2. This is compatible with the Froissart bound, but only for on-shell particles, provided $f(-m_P^2) = 0$. It is shown that there is experimental evidence supporting such behavior.

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Let $\sigma_P(Q^2,s)$ be the total cross section for scattering on some target of the off-shell projectile P (say, a meson) with (unphysical) mass $-Q^2$; s is the square of the c.m. energy. Usually, the Froissart bound is assumed for σ, and hence its high-energy behavior is taken to be controlled by the Pomeron. However, all proofs of the Froissart bound require unitarity[1] and therefore there is no reason why it would hold away from the mass shell. In this Letter we will argue that quantum chromodynamics (QCD) strongly suggest a behavior of the type

$$\sigma_P(Q^2,s) \underset{s\to\infty}{\sim} C_1 f(Q^2) s^\lambda + C_{\rm Pom}(Q^2), \tag{1a}$$

where λ is strictly positive and where, for sufficiently *large* Q^2 that perturbation theory be applicable,

$$f(Q^2) \simeq [\alpha_c(Q^2)]^{-d_+(\lambda)}; \tag{1b}$$