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Spin $\frac{1}{2}$ from Gravity

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For a certain class of three-manifolds, the angular momentum of an asymptotically flat quantum gravitational field can have half-integral values.

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There is a common expectation that at sufficiently small distances the topology of spacetime is not Minkowskian.[1,2] Because such configurations cannot be reached by perturbing flat space, it is by no means obvious that the spin-2 character of linearized gravity will persist nor that the quantum gravitational field must in general have integral spin; and the possibility of spinorial manifolds was in fact suggested more than two decades ago by Finkelstein and Misner.[3] The present work may be regarded as a confirmation of their conjecture that gravity by itself can exhibit half-integral spin. We find in particular that the possible topologies of three-manifolds fall into two classes, those (including R^3, the topologically trivial space) which allow only integral spin, and those which give rise to a space of state vectors having both half-integral and integral spin sectors.

Our construction is somewhat analogous to the appearance of half-integral spin in the quantum mechanics of systems containing both magnetic and electric charges and in the constructions by Jackiw and Rebbi[4] and Hasenfratz and 't Hooft[5] of half-integral spin solitons from coupled Yang-Mills-Higgs and isospinor fields: In these cases and for us, the emergence of spin $\frac{1}{2}$ depends on a configuration space which, because of a gauge degree of freedom, is larger than the space of physical configurations, and on which a 2π rotation can therefore act nontrivially. A basic *difference* is that gravitational spin $\frac{1}{2}$ requires neither source fields nor charges [neither isospinors nor U(1) spinors].

We will work in the Schrödinger or "superspace" picture[6] (This is unrelated to supergravity: Superspace is a space of three-metrics in which one regards as equivalent metrics that differ only by a diffeomorphism, i.e., whose components are related by a coordinate transformation), taking as elements of configuration space asymptotically flat positive-definite metrics g_{ab} on a fixed three-manifold M. The Schrödinger state vector ψ is then a functional on this space of metrics. We begin by describing the kinematics of a theory of quantum gravity on an asymptotically flat space: We introduce the constraint equations and note that they are equivalent to demanding invariance of the wave function ψ under asymptotically trivial diffeomorphisms which are in the component of the identity. Next, we recall the classical Arnowitt-Deser-Misner (ADM) angular momentum[7] and obtain the corresponding quantum operators. We thereby acquire a representation of the rotation group SO(3) [or of its covering group SU(2)] on the space of wave functions, ψ, and so can formally ask whether half-integral spin occurs. We find that for Euclidean topology ($M = R^3$), the representation of SO(3) is single valued: Only integral spin occurs. For other topologies, the key question turns out to be whether a 2π twist of a neighborhood of infinity can be deformed to the identity diffeomorphism —intuitively, whether one can communicate a rotation by 2π at infinity to the whole interior of the space. Remarkably, precisely this question has been addressed and, within the last two years, answered in the context of differential topology[8]; and we easily find examples of topologies for which the representation of SO(3) is double valued and whose associated state space includes subspaces of half-integral angular momentum. Finally, we point out that the crucial 2π rotation whose nontriviality is the criterion for the occurrence of half-integral spin has the meaning of a rotation of the system *relative to its environment* if an asymptotically flat metric is interpreted as representing an isolated system embedded in a larger universe.

Let M be a three-manifold without boundary which is topologically Euclidean outside a compact region. Introduce on a neighborhood N of infinity a strictly Euclidean metric δ_{ab}, and denote by $\mathfrak{M}$ the space of smooth metrics g_{ab} on M which approach δ_{ab} asymptotically.[9] As mentioned above, the state space consists of functions ψ on $\mathfrak{M}$. The generalized position operator $\hat{g}_{ab}$ is defined by[6]

$$\hat{g}_{ab}\psi(g) = g_{ab}\psi(g)$$

and its conjugate momentum operator $\hat{\pi}^{ab}$ is defined by

$$\hat{\pi}^{ab}\psi(g) = -i\delta\psi(g)/\delta g_{ab}.$$

The "physical subspace" $\mathcal{K}$ comprises those ψ satisfying the (momentum) constraint equation

$$D_a\hat{\pi}^{ab}\psi = 0, \tag{1}$$

where D_a is the covariant derivative with respect to g_{ab}. In particular,

$$\int \xi_b D_a \hat{\pi}^{ab}\psi\, dx = 0, \tag{2}$$

for any test field ξ_a of compact support. One requires[10] that the factor ordering give Eq. (2) the meaning

$$0 = \int D_{(a}\xi_{b)}\frac{\delta}{\delta g_{ab}}\psi\, dx = \frac{d}{d\lambda}\psi(g_{ab} + \lambda D_{(a}\xi_{b)})|_{\lambda=0} \tag{3}$$

for all smooth vector fields ξ_a of compact support. One expects that (2) and (3) will hold as well for a larger class of test fields ξ_a vanishing sufficiently rapidly at infinity (for example, the tempered distributions used in flat-space quantum field theory); however, we will need only that all fields of compact support are among the allowed test fields. Equation (3) is equivalent to the statement that $(d/d\lambda)\psi \circ T_\lambda^* = 0$, where T_λ is the one-parameter group of diffeomorphism generated by ξ_a and T_λ^* is the induced action of T_λ on g_{ab}.

Let $\mathfrak{D} = \mathfrak{D}(\mathfrak{M})$ be the class of all diffeomorphisms on M generated by such T_λ ($\mathfrak{D}$ thus consists of asymptotically trivial diffeomorphisms) and let $\mathfrak{D}_0(M)$ be the component of the identity in $\mathfrak{D}$. A functional ψ satisfies (4) iff it is invariant under $T \in \mathfrak{D}_0$:

$$\psi \circ T^* = \psi. \tag{4}$$

Thus the momentum constraint singles out the *connected component*, $\mathfrak{D}_0$, as the natural gauge group of the theory: We can regard as equivalent any two metrics g_{ab} which differ by an action of $\mathfrak{D}_0$ and we call the space of resulting equivalence classes, $\tilde{\mathfrak{M}}$, a superspace of metrics on M. Then the momentum constraint lets us regard ψ as a function on $\tilde{\mathfrak{M}}$. In general, however, ψ will not be invariant under transformations that remain finite at infinity—rotations, for example—nor under transformations such as reflections which are not connected to the identity. [In addition to the constraint (1), ψ must satisfy the Hamiltonian equation

$$\{\hat{g}R + [\tfrac{1}{2}(\hat{\pi}_a{}^a)^2 - \hat{\pi}^{ab}\hat{\pi}_{ab}]\}\psi = 0,$$

which in quantum gravity plays the role of a dynamical equation. This rather intractable equation is not, however, directly involved in our discussion of spin.]

In order to define angular momentum, let φ_α, $\alpha = 1, 2, 3$, be three vector fields on M which on N satisfy the commutation relations

$$[\varphi_\alpha, \varphi_\beta] = -\epsilon_{\alpha\beta\gamma}\varphi_\gamma \tag{5}$$

and preserve the flat metric δ_{ab}. In other words, the φ_α generate on N an isometric realization of SO(3). Then the classical (ADM) angular momentum[7] corresponding to an initial data set (g_{ab}, π^{ab}) has components

$$\begin{aligned} J_\alpha &= -\int_M \mathfrak{L}_{\varphi_\alpha}(g_{ab})\pi^{ab} d^3x \\ &= \lim_{r\to\infty}\left(-2\int \varphi_{\alpha a}\pi^{ab} d\sigma_b\right). \end{aligned} \tag{6}$$

Here π^{ab} is a tensor density, related to the extrinsic curvature K^{ab} by

$$\pi^{ab} = (16\pi)^{-1}(-K^{ab} + g^{ab}K)\sqrt{g},$$

and the second equality uses the classical form of Eq. (1), $D_a\pi^{ab}=0$. It is clear from (6) that the definition of J_α is independent of how the φ_α are extended from N to the "interior" region $K=M-N$.

The corresponding quantum operators $\hat{J}_\alpha$ are defined by

$$\hat{J}_\alpha\psi(g) = -\int_M \mathfrak{L}_{\varphi_\alpha}(g_{ab})\frac{1}{i}\frac{\delta}{\delta g_{ab}}\psi(g)dx$$
$$=\frac{1}{i}\frac{d}{d\theta}\psi\circ R(\theta)^*(g)|_{\theta=0}, \tag{7}$$

where $R_\alpha(\theta)$ is the one-parameter group of diffeomorphisms generated by φ_α. Since any $\psi\in\mathcal{H}$ is invariant under diffeomorphisms $T\in\mathfrak{D}_0$,

$$\psi\circ(R_\alpha\circ T)^* = \psi\circ R_\alpha^*.$$

Consequently, $\hat{J}_\alpha$ is well defined as an operator on $\mathcal{H}$, and, for $\psi\in\mathcal{H}$, $\hat{J}_\alpha\psi$ depends only on the behavior in N of φ_α. The commutator is

$$[\hat{J}_\alpha,\hat{J}_\beta]\psi = (\partial/\partial\zeta)\psi\circ\tilde{R}(\zeta)^*|_{\zeta=0}, \tag{8}$$

where $\tilde{R}(\zeta)$ is the family of diffeomorphisms generated by $-[\varphi_\alpha,\varphi_\beta]$. But (8) is again independent of $\tilde{R}$ outside N, and in N, where (5) holds, we have $\tilde{R}=R_\gamma$ (α,β,γ cyclic). Thus on $\mathcal{H}$,

$$[\hat{J}_\alpha,\hat{J}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{J}_\gamma. \tag{9}$$

A state $\psi\in\mathcal{H}$ has half-integral angular momentum iff (with $J\equiv J_\alpha$, $R\equiv R_\alpha$ for any choice of α) the rotated state vector,

$$e^{2\pi i\hat{J}}\psi = \psi\circ R(2\pi)^*, \tag{10}$$

is $-\psi$. Now if $R(2\pi)\in\mathfrak{D}_0$, then (4) and (10) imply that $e^{2\pi i\hat{J}}\psi=\psi$ for all $\psi\in\mathcal{H}$. On the other hand, if $R(2\ )$ is not in $\mathfrak{D}_0$, then there will be wave functions $\psi\in\mathcal{H}$ for which $\psi'=\psi\circ R(2\pi)^*\neq\psi$. Since in $\mathcal{H}$, $e^{4\pi i\hat{J}}=1$ (and since $\psi'\in\mathcal{H}$), the difference $\psi'-\psi$ will be an eigenstate in $\mathcal{H}$ of $e^{2\pi i\hat{J}}$ with eigenvalue -1. Let us call M "spinorial" when states of half-integral angular momentum occur in $\mathcal{H}(M)$. We conclude that *M is spinorial* iff *the 2π twist $R(2\pi)$ is not in* $\mathfrak{D}_0$, i.e., iff it is not isotopic to the identify in the group $\mathfrak{D}$ of asymptotically trivial diffeomorphisms.

When M is topologically Euclidean, it is easy to see that half-integral spin cannot occur. For then we can take for N the whole manifold M and, for the φ_α, fields which *everywhere* satisfy (5), so that $R(2\pi)=1$.

In the general case, by using our freedom to alter φ^a (φ^a refers to any $\varphi_\alpha{}^a$) in a compact set, we can replace φ^a by

$$\tilde{\varphi}^a = \begin{cases} 0, & x\in N \\ f(r)\varphi^a(x), & x\in N, \end{cases}$$

where r is the radial coordinate[11] on N and $f(r)$ rises smoothly from 0 at some r_1 to 1 at some $r_2>r_1$. With $\tilde{\varphi}^a$ in this form $\tilde{R}(2\pi)=\exp(2\pi\tilde{\varphi}^a)$ is the identity on K and also on N for $r<r_1$, $r>r_2$; for $r_1<r<r_2$, $\tilde{R}(2\pi)$ rotates the spheres $r=$constant through an angle between 0 and 2π. The diffeomorphism $\tilde{R}(2\pi)$ is called by Hendriks[8] a rotation parallel to a sphere, and from his work we infer the following key result: $R(2\pi)$ is in $\mathfrak{D}_0$ iff M is a connected sum[12]

$$M=R^3\# M_1\#\ldots\# M_k$$

of compact three-manifolds (without boundary) each of which (i) is homotopic to $P^2\times S^1$ (P^2 is the real projective two-sphere), or (ii) is homotopic to an S^2 fiber bundle over S^1, or (iii) has a finite fundamental group $\pi_1(M_j)$ whose two-Sylow subgroup[13] is cyclic.

Given this criterion, it is easy to find compact three-manifolds N for which the manifold $M=R^3\# N$ [equivalently, N with a point (the point at infinity) removed] is spinorial. One example is the three-torus $T^3=S^1\times S^1\times S^1$ whose fundamental group $(Z\times Z\times Z)$ is infinite. We can get other examples from finite subgroups G of SU(2). Because SU(2) is topologically a three-sphere, the manifold of cosets $N=\mathrm{SU}(2)/G$ has $\pi_1(N)=G$. Hence $R^3\# N$ will be spinorial iff G has a noncyclic two-Sylow subgroup. Moreover, since N is by definition an "elliptic" space, M occurs classically as a spacelike hypersurface of an asymptotically flat vacuum space-time (in contrast to $R^3\# T^3$ which seems to be classically forbodden).[13,14] One G that works is $Q=(\pm 1,\pm i\sigma_1,\pm i\sigma_2,\pm i\sigma_3)$, which, being of order $8=2^3$, is its own two-Sylow subgroup. One can also construct the resulting M by removing from R^3 a solid cube and identifying opposite faces of its boundary with a $90°$ rotation.

Our discussion so far has presupposed an asymptotically flat space-time and interpreted angular momentum in terms of the symmetry group at spatial infinity. Physically, such a space-time is simply a representation of an isolated system, which abstracts from the details of the system's environment. But rotation of a system with respect to a larger universe in which it is embedded makes sense, and we want to show that $R(\theta)$ can be so understood. One could imagine, for example, cutting out of a space a piece containing a handle, rotating it, and then gluing it back in, and so ending with a differently oriented handle. This process will be well defined if there is a spherically symmetric interface of finite thickness joining the handle to the geometry outside. And such an interface can be regarded as the appropriate limit of a realistic interface which is far from a microscopic handle and at the same time is small enough that the large-scale curvature of the background space can be neglected. Only in such a situation is one entitled to regard the handle as an isolated system (from afar, as a particle), to model it by an asymptotically flat space-time, and to ascribe to it a "spin" of its own.

Given then a manifold U (universe) divided into an interior region M and an exterior region E such that $M \cap E$ is a spherically symmetric thick shell, a "rotation by θ of M with respect to E" is a three-geometry obtained by cutting M at any sphere $S \subset M \cap E$ and reidentifying after rotating the inner piece by θ with respect to the outer. Since each manifold so obtained is diffeomorphic to U, a sequence $0 \leq \theta \leq 2\pi$ of such rotated three-geometries is equally a sequence of three-metrics g_{ab} on the fixed manifold M. The final three-metric in this sequence is diffeomorphic to the original, but the diffeomorphism is a rotation parallel to the sphere S! Thus the spinorial manifolds M are those for which a continuous 2π rotation of M with respect to a generic environment is not deformable to the zero rotation. We can also grasp from the present standpoint why, for example, overall rotations of M were not in the gauge group $\mathfrak{D}(M)$: Here they effect a real change in the *relation* of M to its environment.

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[11]The radial coordinate r indexes metric spheres of δ_{ab} invariant under the action of $\{\phi_\alpha\}$.

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[13]A two-Sylow subgroup of a finite group G is a subgroup whose order is a power of 2 (possibly 2^0) and which is properly contained in no larger such subgroup. All two-Sylow subgroups of a given group are isomorphic.

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