Pauli Principle and the Optical Potential for Elastic Two-Fragment Collisions

R. Goldflam and K. L. Kowalski
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
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The Pauli principle is introduced into the theory of the optical potential in a consistent manner. This is facilitated by an appropriate choice of off-shell extension for the two-fragment transition operators. Dynamical equations for the optical potential are derived and low-order approximations are discussed.

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The device of a complex, equivalent one-particle optical potential (OP) is widely used to describe the elastic scattering of two nuclear fragments. However, the microscopic theory of the OP is deficient in providing both a consistent method for the calculation of this object and a practical inclusion of the Pauli principle.[1] A consistent nonrelativistic theory of the OP has been proposed recently in the case where all the particles are distinguishable.[2] In this Letter we outline how that work can be generalized to include the effects of particle identity.[3] The central idea of our development is the recognition of the constraints upon the off-shell structure of the transition operators for rearrangement scattering into physically indistinguishable channels imposed by the demand of the existence of an OP. These constraints are essentially formalism independent and thus are directly relevant to approaches to the theory of the OP besides the one considered here.

The amplitude for the elastic scattering of two composite fragments is given by

$$\langle \varphi_{\overline{\beta}}(\vec{k}_f)|T(\hat{\beta})|\varphi_{\overline{\beta}}(\vec{k}_i)\rangle,$$

where $|\varphi_{\overline{\beta}}(\vec{k})\rangle$ is the two-fragment ground state with relative momentum $\vec{k}$ and $T(\hat{\beta})$ is the fully symmetrized transition operator[4,5]

$$T(\hat{\beta})=\sum_{\beta}\delta(\beta\in\hat{\beta})\mathfrak{A}_{\beta,\overline{\beta}}{}^{\dagger}T_{\beta,\overline{\beta}}. \qquad (1)$$

In (1) $T_{\beta,\overline{\beta}}$ can be taken to be equal to the usual unsymmetrized transition operator $T_{\beta,\overline{\beta}}{}^{(+)}=V^{\beta}G\times G_{\overline{\beta}}{}^{-1}$ to within terms which vanish on shell. A canonical[4] two-cluster partition is denoted by $\overline{\beta}$, while $\hat{\beta}$ is the set of all partitions which can be obtained from $\overline{\beta}$ by permutations. $\delta(\beta\in\hat{\beta})=1$ if $\beta\in\hat{\beta}$ and $\delta(\beta\in\hat{\beta})=0$ if $\beta\notin\hat{\beta}$.

The optical potential $U(\hat{\beta})$ is defined by

$$T(\hat{\beta})=U(\hat{\beta})+U(\hat{\beta})g_{\overline{\beta}}T(\hat{\beta}), \qquad (2)$$

where $g_{\gamma}=G_{\gamma}P_{\gamma}$ and P_{γ} is the projector onto the space spanned by $\{|\varphi_{\gamma}(\vec{k})\rangle\}$. The equivalent one-body OP in momentum space is $\mathcal{V}_{\hat{\beta}}=\langle\varphi_{\overline{\beta}}(\vec{k}')|U(\hat{\beta})\times|\varphi_{\overline{\beta}}(\vec{k})\rangle$. Definition (2) takes on content when it is supplemented by a symmetrized dynamical scattering integral equation[4]

$$T(\hat{\beta})=A(\hat{\beta})+K(\hat{\beta})T(\hat{\beta}), \qquad (3)$$

where $A(\hat{\beta})$ and $K(\hat{\beta})$ are determined by the microscopic interactions. From (2) and (3) we then obtain a dynamical equation for $U(\hat{\beta})$:

$$U(\hat{\beta})=A(\hat{\beta})+[K(\hat{\beta})-A(\hat{\beta})g_{\overline{\beta}}]U(\hat{\beta}). \qquad (4)$$

Obviously different choices for the off-shell extensions of the operators $T_{\beta,\overline{\beta}}$ which represent rearrangements into physically indistinguishable channels β lead, via (1)–(4), to different potentials $\mathcal{V}_{\hat{\beta}}$. We constrain these choices by requiring that $\mathcal{V}_{\hat{\beta}}$ be real except for the effects of inelasticities[6] and that it also be independent of the choice of the canonical partition $\overline{\beta}$.

If the off-shell extension of $T(\hat{\beta})$ is such that both $A(\hat{\beta})$ and $K(\hat{\beta})-A(\hat{\beta})g_{\overline{\beta}}$ are continuous across the elastic unitarity cuts, then $U(\hat{\beta})$ shares this property and our first requirement is satisfied. This indicates how the reality properties[6] of the OP are controlled by the off-shell behavior of $T(\hat{\beta})$, but it gives us no indication as to how a proper choice of this behavior is to be made. A hint can be drawn from the fact that the reality properties of $\mathcal{V}_{\hat{\beta}}$ depend on the off-shell unitarity relations satisfied by $T(\hat{\beta})$. With the choice $T^{(+)}(\hat{\beta})$ obtained with $T_{\beta,\overline{\beta}}{}^{(+)}$ in (1), e.g., asymmetries appear in these relations which are inconsistent with (2) and a properly behaved OP. Similar remarks apply to the choice $T_{\beta,\overline{\beta}}{}^{(-)}=G_{\beta}{}^{-1}GV^{\overline{\beta}}$.

The preceding asymmetries result from the inequivalent roles of β and $\overline{\beta}$ in the behavior of $T_{\beta,\overline{\beta}}{}^{(+)}$ off shell. This inequivalence would also appear to introduce a spurious dependence in $\mathcal{V}_{\hat{\beta}}$ upon the canonical partition $\overline{\beta}$ which violates our second requirement on the OP. This suggests that we choose a set of transition operators $T_{\beta,\overline{\beta}}$

which are related to the residues of G nonpreferentially with respect to $\overline{\beta}$ and β, e.g., the operators introduced by Alt, Grassberger, and Sandhas (AGS)[7]

$$\overline{T}_{\beta,\overline{\beta}} = T_{\beta,\overline{\beta}}^{(+)} + \overline{\delta}_{\beta,\overline{\beta}} G_{\overline{\beta}}^{-1}, \tag{5}$$

where $\overline{\delta}_{\beta,\overline{\beta}} = 1 - \delta_{\beta,\overline{\beta}}$. We remark that below we use (5) with β and $\overline{\beta}$ replaced by arbitrary partitions a and b; also, $\overline{T}_{a,b} \neq \overline{T}_{b,a}$. With the methods of Ref. 2 we show that the choice (5) yields a $\mathcal{V}_{\hat{\beta}}$ with our stipulated properties.

The operator $\tau^{\alpha,\beta} \equiv T_{\alpha,\beta}^{(+)} - V_\beta^\alpha$ satisfies the set of connected-kernel integral equations[2,8]

$$\tau^{\alpha,\beta} = \sum_a{}' W^{\alpha,\beta}(a) + \sum_\gamma W^{\alpha,0}(\gamma) G_0 \tau^{\gamma,\beta}, \tag{6}$$

where $W^{b,c}(a)$ is the a-connected part of $\tau^{b,c}$ and a,b,c are general partitions.[2] The prime on the sum in (6) denotes omission of the fully connected term. G_0 is the free Green's function and the subscript 0 refers to the N-cluster partition. The major advantage of (6) is the explicit multiple-scattering structure which resides in its inhomogeneous term. For example, $W^{\alpha,\beta}(i) = t_i$, where t_i is the ordinary two-particle transition operator and i is a pair label [$(N-1)$-cluster partition] external to α and β.

The derivation of a dynamical equation for $U(\hat{\beta})$ is based on the subtraction of the discontinuities across all the elastic unitarity cuts in the class $\hat{\beta}$. To this end we introduce the operators $\Lambda_\lambda{}^\alpha(\hat{\beta})$ as solutions of the connected-kernel integral equations

$$\Lambda_\lambda{}^\alpha(\hat{\beta}) = \delta_{\alpha,\lambda} + \sum_\gamma [W^{\alpha,0}(\gamma) G_0 - V_\gamma{}^\alpha g_\gamma \delta(\gamma \in \hat{\beta})] \Lambda_\lambda{}^\gamma(\hat{\beta}), \tag{7a}$$

$$\Lambda_\lambda{}^\alpha(\hat{\beta}) = \delta_{\alpha,\lambda} + \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) [W^{\gamma,0}(\lambda) G_0 - V_\lambda{}^\gamma g_\lambda \delta(\lambda \in \hat{\beta})], \tag{7b}$$

whose kernels are continuous across the $\hat{\beta}$-class unitarity cuts.[2,9] The combination of (6) and (7) yields

$$\tau^{\alpha,\beta} = \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) \sum_a{}' W^{\gamma,\beta}(a) + \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) \sum_\lambda \delta(\lambda \in \hat{\beta}) V_\lambda{}^\gamma g_\lambda \tau^{\lambda,\beta}. \tag{8}$$

If we take $\alpha, \beta \in \hat{\beta}$ in (8) and then use (5) for $\overline{T}_{\alpha,\beta}$ we find with the aid of (B15) and (7b) that[8]

$$\overline{T}_{\alpha,\beta} = \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) I_\beta{}^\gamma(\hat{\beta}) + \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) \sum_\lambda \delta(\lambda \in \hat{\beta}) V_\lambda{}^\gamma g_\lambda \overline{T}_{\lambda,\beta}, \tag{9}$$

where

$$I_\beta{}^\gamma(\hat{\beta}) = \{[\sum_{a \notin \hat{\beta}}{}' W^{\gamma,0}(a) + W^{\gamma,0}(\beta)] G_0 + \overline{\delta}_{\gamma,\beta} \delta(\gamma \in \hat{\beta})\} \delta(\beta \in \hat{\beta}) G_\beta{}^{-1}. \tag{10}$$

Since $I_\beta{}^\gamma(\hat{\beta})$ has no $\hat{\beta}$-class elastic unitarity singularities[9] we see that such singularities appear only in the kernels of (9). Similar properties hold for the form of (9) which results if (5) is *first* used in (6) with arbitrary α and β and then the resultant equation is combined with (B15) and (7). The inhomogeneous term of (9) can be written in several alternative ways[9] with (7b) and (B15).

We next introduce a set of operators $\mathfrak{U}_{\alpha,\beta}(\hat{\beta})$ defined by $(\alpha, \beta \in \hat{\beta})$

$$\mathfrak{U}_{\alpha,\beta}(\hat{\beta}) = \overline{T}_{\alpha,\beta} - \sum_\lambda \delta(\lambda \in \hat{\beta}) \overline{T}_{\alpha,\lambda} g_\lambda \mathfrak{U}_{\lambda,\beta}(\hat{\beta}). \tag{11}$$

From (9) and (11) we infer that

$$\mathfrak{U}_{\alpha,\beta}(\hat{\beta}) = \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) I_\beta{}^\gamma(\hat{\beta}) - \sum_\lambda \delta(\lambda \in \hat{\beta}) K_\lambda{}^\alpha(\hat{\beta}) \mathfrak{U}_{\lambda,\beta}(\hat{\beta}), \tag{12}$$

where the kernel of (12) is, for $\alpha, \lambda \in \hat{\beta}$,

$$K_\lambda{}^\alpha(\hat{\beta}) \equiv [\sum_\beta \delta(\beta \in \hat{\beta}) \Lambda_\beta{}^\alpha(\hat{\beta}) - \delta_{\alpha,\lambda} + \sum_\gamma \Lambda_\gamma{}^\alpha(\hat{\beta}) \sum_{a \notin \hat{\beta}} W^{\gamma,0}(a) G_0] P_\lambda, \tag{13}$$

and is also free of $\hat{\beta}$-class unitarity singularities.

It can be shown that all of the operators appearing in Eqs. (5)–(13) are, in the terminology of Ref. 4, *label transforming* under the symmetry transformations which represent permutations of the nucleons. This means that under such a transformation an operator is mapped into itself but with any of its partition labels replaced by the partitions which result from the permutation mappings of the original labels. We can then apply the techniques of Ref. 4 in a straightforward manner to obtain integral equations for the symmetrized operators $T(\hat{\beta})$ and $U(\hat{\beta})$. Specifically, from (11) we have

$$\overline{T}_{\alpha,\beta} = \mathfrak{U}_{\alpha,\beta}(\hat{\beta}) + \sum_\lambda \delta(\lambda \in \hat{\beta}) \mathfrak{U}_{\alpha,\lambda}(\hat{\beta}) g_\lambda \overline{T}_{\lambda,\beta}, \tag{14}$$

and so we recover Eq. (2) with

$$U(\hat{\beta}) = \sum_\beta \delta(\beta \in \hat{\beta}) \mathcal{A}_{\beta,\bar{\beta}}^\dagger U_{\beta,\bar{\beta}}(\hat{\beta}). \tag{15}$$

Finally, from (12) we find [cf. Eq. (4)]

$$U(\hat{\beta}) = [\sum_\beta \delta(\beta \in \hat{\beta}) \mathcal{A}_{\beta,\bar{\beta}}^\dagger \sum_\gamma \Lambda_\gamma{}^\beta(\hat{\beta}) I_{\bar{\beta}}{}^\gamma(\hat{\beta})] - [\sum_\beta \delta(\beta \in \hat{\beta}) \mathcal{A}_{\beta,\bar{\beta}}^\dagger K_{\bar{\beta}}{}^\beta(\hat{\beta})] U(\hat{\beta}). \tag{16}$$

Equation (16) is a dynamical integral equation for the optical-potential operator $U(\hat{\beta})$ whose driving and kernel terms are free of all $\hat{\beta}$-class elastic unitarity singularities and which includes all effects of the Pauli principle. It is easily verified that the kernel of (16) is a connected operator. Also the input into the kernel and driving terms of (16) consists entirely of subsystem dynamics [the $W^{\gamma,0}(a)$ operators] and the solutions $\Lambda_\gamma{}^\beta(\hat{\beta})$ of the connected-kernel integral Eqs. (7). Thus, (16) along with (7) constitutes a consistent theory of the optical potential in that $U(\hat{\beta})$ can be calculated in a well-defined manner, at least in principle. This also allows the systematic calculation of corrections to low-order approximations. As long as these approximations leave the operators (16) label transforming, the *matrix elements* $\mathcal{V}_{\hat{\beta}}(\vec{k}'|\vec{k})$ are independent of the choice of $\bar{\beta}$. We remark that (16) reduces to a one-dimensional integral equation for the partial-wave amplitudes of $\mathcal{V}_{\hat{\beta}}(\vec{k}'|\vec{k})$.

Our results permit the reconsideration and extension of all the standard approximations to the OP but, more significantly, we are presented with the opportunity to construct entirely new calculational methods. These questions will be pursued elsewhere. It is instructive here, however, to examine a low-order approximation to (16) after rewriting its inhomogeneous term using (B15) and (7b). In the resultant form of (16) we neglect all $W^{a,b}(c)$ except for $c = i$, take $\Lambda_\gamma{}^\beta(\hat{\beta}) \simeq \delta_{\beta,\gamma}$, and drop all terms involving $V_\gamma{}^\beta P_\gamma$. Then (16) becomes

$$U(\hat{\beta}) = \sum_{\beta \in \hat{\beta}} \mathcal{A}_{\beta,\bar{\beta}}^\dagger (\sum_i \bar{\Delta}_{\beta,i} t_i \bar{\Delta}_{\bar{\beta},i} + \bar{\delta}_{\beta,\bar{\beta}} G_{\bar{\beta}}^{-1} + V_{\bar{\beta}}{}^\beta) - \sum_{\beta \in \hat{\beta}} \mathcal{A}_{\beta,\bar{\beta}}^\dagger (\sum_i \bar{\Delta}_{\beta,i} t_i G_0 + \bar{\delta}_{\beta,\bar{\beta}}) P_{\bar{\beta}} U(\hat{\beta}), \tag{17}$$

where $\bar{\Delta}_{\beta,i} = 1$ if i is external to β and vanishes otherwise. Calculations made with the use of (17) are well within present limits of practicality even though (17) represents a considerable generalization of the usual antisymmetrized impulse approximation which results from the $\bar{\Delta}_{\beta,i} t_i \bar{\Delta}_{\bar{\beta},i}$ terms and yields "$t\rho$"-type amplitudes. We recall that t_i is a *two-particle operator* and so the input into (17) contains no hidden N-particle dynamics. Additional approximations to (17) which appear to be consistent with a low-density limit are the neglect of the nonorthogonality terms proportional to $\bar{\delta}_{\beta,\bar{\beta}}$ and those t_i's in the kernel with i internal to $\bar{\beta}$.

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[1]*Microscopic Optical Potentials*, edited by H. V. von Geramb (Springer, Berlin, 1979).

[2]K. L. Kowalski, Ann. Phys. (N. Y.) 120, 328 (1979).

[3]For simplicity we regard all nucleons as identical. We also confine ourselves to only pairwise nucleon-nucleon interactions.

[4]G. Bencze and E. F. Redish, J. Math. Phys. 19, 1909 (1978).

[5]The partitions of the N nucleons into two fragments are denoted by α, β, etc. The channel and full Green's functions are $G_\beta = (E - H_\beta + i0)^{-1}$ and $G = (E - H + i0)^{-1}$, respectively. V^β contains all interactions external to the clusters of β. $\mathcal{A}_{\beta,\bar{\beta}}$ is the product of the parity-weighted unitary operator which represents the permutations $\bar{\beta} \to \beta$ and the antisymmetrizer with respect to all permutations which map $\bar{\beta}$ into itself, $\mathcal{A}(\bar{\beta})$. Note $\mathcal{A}(\bar{\beta})|\varphi_{\bar{\beta}}\rangle = |\varphi_{\bar{\beta}}\rangle$. Below we use $V_\alpha{}^\beta$ which contains all the interactions internal to α and external to β.

[6]We mean by this the continuity of $U(\hat{\beta})$ across all of the elastic unitarity cuts associated with the physically equivalent channels $\beta \in \hat{\beta}$.

[7]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967); P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967).

[8]The inhomogeneous term of (6) is equal to

$$\Sigma_a{}' W^{\alpha,0}(a) G_0 G_\beta{}^{-1} + \Sigma_\gamma W^{\lambda,0}(\gamma) G_0 V_\beta{}^\gamma - V_\beta{}^\alpha.$$

We call this identity (B15) (cf. Ref. 2).

[9]The term $V_\gamma{}^\alpha g_\gamma$ contains the γ-elastic unitarity singularity of $W^{\alpha,0}(\gamma) G_0$ so that the difference of these operators is continuous across the γ-elastic unitary cut and, thus, so is the operator $W^{\alpha,0}(\gamma) G_0 G_\gamma{}^{-1}$.