Filamentation Instability of Electron and Positron Colliding Beams in Storage Ring

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The filamentation instability of the electron and positron colliding beams in a storage ring is investigated within the framework of the Vlasov-Maxwell equations, and a closed algebraic dispersion relation for the complex eigenfrequency ω is obtained. It is shown that the typical growth rate of instability is a substantial fraction of the electron plasma frequency ω_{pe}. For parameters characterizing the recent colliding-beam experiments at DESY, the instability threshold is marginally satisfied.

There is a growing interest in the stability properties of the electron-positron colliding beams in a storage-ring facility.[1,2] A recent experiment with colliding electron-positron beams at DESY has shown the broadening of the beam cross section, suggesting an instability of collective modes which may limit the achievable intensity.[3] Perhaps one of the most important instabilities of the electron and positron colliding beams in a storage ring is the filamentation instability. The unstable modes propogate nearly perpendicular to the beam with mixed electrostatic and electromagnetic components, the latter destabilizing and the former stabilizing. The perturbed magnetic field is mostly in the plane perpendicular to the beam and the Lorentz force causes the beam to filamentate, similar to the Weibel instability. Unlike the Weibel modes, which are purely electromagnetic for counterstreaming electron beams, the linear perturbations of colliding electron-positron beams cause both charge and current perturbations giving rise to mixed polarizations. Furthermore, for the case of colliding beams with radial dimension smaller than the collisionless skin depth c/ω_p, the finite geometry becomes important and the usual assumption of an infinite, homogeneous medium is no longer valid.

In this Letter, we treat the filamentation instability of colliding electron-positron beams with finite-geometry effects included. For simplicity, we assume that this colliding beam is straight and infinite along the axial direction. The analysis is carried out within the framework of the Vlasov-Maxwell equations. An important conclusion of the present analysis is that the typical growth rate of the filamentation instability is of the order of the electron plasma frequency ω_{pe}, thereby severely limiting the electron density in a storage ring.

As illustrated in Fig. 1, the equilibrium configuration consists of intense relativistic electron and positron beams propagating opposite to each other with axial velocity $\beta_p c\hat{e}_z$ for the positron beam and $\beta_e c\hat{e}_z$ for the electron beam, where $\hat{e}_z$ is a unit vector along the z direction, c is the speed of light *in vacuo*, and $\beta_p = -\beta_e$. Moreover, both beams have the same radius R_b and the same characteristic energy $\gamma_b mc^2$. It is also assumed that the ratio of the beam radius to the collisionless skin depth c/ω_p is small, i.e.,

$$\frac{\nu_j}{\gamma_b} = N_j \frac{e^2}{mc^2}\frac{1}{\gamma_b} \ll 1, \tag{1}$$

where $j=e,p$ denote electrons and positrons, respectively, ν_j is Budker's parameter, $N_j = 2\pi \times \int_0^\infty dr\, r n_j^0(r)$ is the number of particles per unit axial length, $n_j^0(r)$ is the equilibrium particle density of beam component j, $-e$ and m are the charge and rest mass, respectively, of electron. As shown in Fig. 1, we introduce a cylindrical

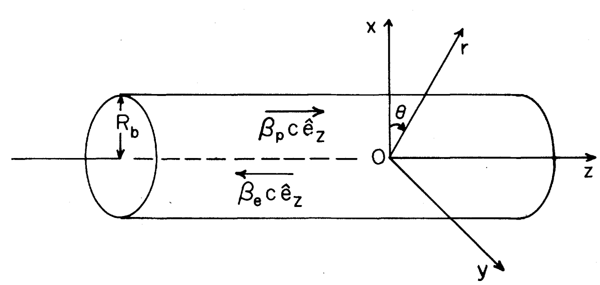

FIG. 1. Equilibrium configuration and coordinate system.

polar coordinate system (r, θ, z). For simplicity, all equilibrium properties are assumed to be azimuthally symmetric $(\partial/\partial\theta = 0)$ and independent of axial coordinate $(\partial/\partial z = 0)$. There are three constants of the motion: the total energy, $H = (m^2c^4 + c^2p^2)^{1/2} + e_j\varphi_0(r)$; the canonical angular momentum, $P_\theta = rp_\theta$; and the axial canonical momentum, $P_z = p_z + (e_j/c)A_z^s(r)$. Here, $\varphi_0(r)$ is the equilibrium self-electric potential, $A_z^s(r)$ is the axial component of vector potential for the azimuthal self-magnetic field, e_j is the charge of particles of beam component j (i.e., $e_j = -e$ for $j = e$ and $e_j = e$ for $j = p$), and $\vec{p}$ denotes mechanical momentum and is related to the particle velocity $\vec{v}$ by $\vec{v} = \vec{p}/[m(1 + p^2/m^2c^2)^{1/2}]$. For beams of well-defined energy and momentum, an equilibrium associated with the rigid-rotor beam-distribution function,[4]

$$f_j^0(\vec{x}, \vec{p}) = \frac{\hat{n}_j}{2\pi\gamma_b m}\,\delta(H - \omega_j P_\theta - \hat{\gamma}_j mc^2)\,\delta(P_z - \gamma_b m\beta_j c), \tag{2}$$

is particularly suited for stability analysis where $\hat{n}_j$ is the particle density at $r = 0$, the constant ω_j and $\hat{\gamma}_j mc^2$ are the rotation frequency and the total energy in the rotating frame of beam component j, respectively.

Since the $r-\theta$ kinetic energy of particles is small in comparison with the characteristic axial energy $\gamma_b mc^2$, it is straightforward to show that the term $H - \omega_j P_\theta$ in Eq. (2) can be approximated by[4,5]

$$H - \omega_j P_\theta = \gamma_b mc^2 + p_\perp^2/2\gamma_b m + \tfrac{1}{2}\gamma_b m\Omega_j^2 r^2, \tag{3}$$

where

$$\gamma_b^2 = (1 - \beta_p^2)^{-1}, \quad p_\perp^2 = p_r^2 + (p_\theta - \gamma_b m\omega_j r)^2,$$

$$\Omega_j^2 = (\hat{\omega}_j - \omega_j)(\omega_j + \hat{\omega}_j) = -\omega_j^2 - (2\pi e_j/\gamma_b m)\textstyle\sum_k \hat{n}_k e_k(1 - \beta_j\beta_k)$$

is the square of the betatron frequency, and the laminar rotation frequency $\hat{\omega}_j$ is defined by

$$\hat{\omega}_j = [(-2\pi e_j/\gamma_b m)\textstyle\sum_k \hat{n}_k e_k(1 - \beta_j\beta_k)]^{1/2}.$$

Here the subscript $k = e, p$.

Substituting Eq. (3) into Eq. (2), we find the equilibrium particle density profile $n_j^0(r) = \hat{n}_j$ for $0 \leq r < R_b$ and $n_j^0(r) = 0$; otherwise, where the beam radius R_b is defined by

$$R_b^2 = 2c^2(\hat{\gamma}_j - \gamma_b)/\gamma_b\Omega_j^2 \tag{4}$$

for $j = e, p$. Equation (4) ensures that the electron and positron beams have the common beam radius R_b. It is important to note from Eq. (4) that the radially confined equilibrium exists only for the rotational frequency ω_j satisfying $-\hat{\omega}_j < \omega_j < \hat{\omega}_j$. Additional equilibrium properties associated with the distribution function in Eq. (2) are discussed in Ref. 4.

We make use of the linearized Vlasov-Maxwell equations to obtain the dispersion relation for filamentation instability of the electron and positron beams. For perturbations with aximuthal harmonic number l and axial wave number k_z, a perturbed quantity $\delta\Phi(\vec{x}, t)$ can be expressed as $\delta\Phi(\vec{x}, t) = \hat{\Phi}(r) \times\exp[i(l\theta + k_z z - \omega t)]$, where ω is the complex eigenfrequency. The present stability analysis is carried out for long parallel wavelength and low-frequency perturbations satisfying $k_z^2R_b^2 \ll l^2$, $|\omega R_b/c|^2 \ll l^2$. With this assumption, the axial components of perturbed fields $E_z(r)$ and $B_z(r)$ are negligible and the Maxwell equations of perturbed potentials can be expressed as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2}\right)\hat{\varphi}(r) = -4\pi\hat{\rho}(r) \tag{5}$$

and

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2}\right)\hat{A}(r) = -\frac{4\pi}{c}\hat{J}_z(r), \tag{6}$$

where $\hat{\varphi}(r)$ is the perturbed electrostatic potential, $\hat{\rho}(r)$ is the perturbed charge density, and $\hat{A}(r)$ and $\hat{J}_z(r)$ are the axial components of the perturbed vector potential and current density, respectively. Components of perturbed fields can be expressed in terms of $\hat{\varphi}(r)$ and $\hat{A}(r)$ as $E_\theta = -il\hat{\varphi}(r)/r$, $\hat{E}_r(r) = -(\partial/\partial r)\hat{\varphi}(r)$, $\hat{B}_r(r) = il\hat{A}(r)/r$, and $\hat{B}_\theta(r) = -(\partial/\partial r)\hat{A}(r)$.

In order to calculate perturbed charge and current densities, we solve the linearized Vlasov equation[5] to obtain the perturbed distribution function

$$\hat{f}_j(r,\vec{p}) = \frac{e_j\gamma_b m}{p_\perp}\frac{\partial f_j^0}{\partial p_\perp}\{\hat{\psi}_j(r) + (\omega - l\omega_j - k_z\beta_j c)\int_{-\infty}^0 d\tau\, i\hat{\psi}_j(r')\exp[il(\theta'-\theta) - i(\omega - k_z\beta_j c)\tau]\}, \quad (7)$$

where the perturbed electrostatic potential $\hat{\psi}_j(r)$ in frame of reference moving with velocity $\beta_j c$ is defined by $\hat{\psi}_j(r) = \hat{\varphi}(r) - \beta_j\hat{A}(r)$ and use has been made of $p_z/\gamma_b m \simeq \beta_j$ consistent with Eq. (1). It is useful to introduce the polar momentum variables $(p_\perp, \varphi)$ in the rotating frame defined by[4,5] $p_x + \gamma_b m\omega_j y = p_\perp\cos\theta$, $p_y - \gamma_b m\omega_j x = p_\perp\sin\varphi$. Note also that the Cartesian coordinates (x,y) are related to the polar coordinates (r,θ) by $x = r\cos\theta$ and $y = r\sin\theta$. In this context, the transverse equation of motion of particles can be expressed as[5]

$$x'(\tau) = (1/\hat{\omega}_j)[(p_\perp/\gamma_b m)\cos\varphi\sin\hat{\omega}_j\tau - r\omega_j\sin\theta\sin\hat{\omega}_j\tau + r\hat{\omega}_j\cos\theta\cos\hat{\omega}_j\tau],$$

$$y'(\tau) = (1/\hat{\omega}_j)[(p_\perp/\gamma_b m)\sin\varphi\sin\hat{\omega}_j\tau + r\omega_j\cos\theta\sin\hat{\omega}_j\tau + r\hat{\omega}_j\sin\theta\cos\hat{\omega}_j\tau], \quad (8)$$

where $\tau = t' - t$, and the harmonic frequency $\hat{\omega}_j$ is defined in Eq. (3).

Upon integration of Eq. (7), the perturbed charge density can be found to be

$$\hat{\rho}(r) = 2\pi e^2\sum_j \gamma_b m\int_0^\infty dp_\perp p_\perp\int_{-\infty}^\infty dp_z\frac{1}{p_\perp}\frac{f_j^0}{p_\perp}[\hat{\psi}_j(r) + (\omega - l\omega_j - k_z\beta_j c)\hat{I}_j], \quad (9)$$

where the orbit integral $\hat{I}_j$ is defined by

$$\hat{I}_j = i\int_0^{2\pi}\frac{d\varphi}{2\pi}\int_{-\infty}^0 d\tau\,\hat{\psi}_j(r')\exp\{i[l(\theta'-\theta) - (\omega - k_z\beta_j c)\tau]\}. \quad (10)$$

Similarly the perturbed axial current density can be obtained. It can be shown in subsequent analysis that Eqs. (5) and (6) with Eq. (9) support a class of solution[5] in which the perturbed charge and current densities are equal to zero except at $r = R_b$. In this regard, we will consider here a class of special solutions for which the perturbed charge and current density are localized on the beam surface. More general perturbations are to be presented in a subsequent publication. In this case it follows from Eqs. (5) and (6) that the function $\psi_j(r)$ has the simple form $\hat{\psi}_j(r) = \hat{\varphi}(r) - \beta_j\hat{A}(r) = C_j r^l$ for $0 < r < R_b$. Here $C_j = d_1 - \beta_j d_2$, and d_1 and d_2 are constants. Substituting Eq. (8) into Eq. (10), it is readily shown that

$$\hat{I}_j = i\psi_j(r)(2\omega_j)^{-l}\int_{-\infty}^0 d\tau\exp[-i(\omega - k_z\beta_j c)\tau][(\omega_j + \hat{\omega}_j)\exp(i\hat{\omega}_j\tau) - (\omega_j - \hat{\omega}_j)\exp(-i\hat{\omega}_j\tau)]^l.$$

After some straightforward algebra that utilizes Eqs. (2) and (13), Eq. (9) can be expressed as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2}\right)\hat{\varphi}(r) = -\sum_j\hat{\psi}_j(r)\frac{\omega_{pj}^2}{\Omega_j^2 R_b}\Gamma_j(\omega)\delta(r - R_b), \quad (11)$$

where $\omega_{pj}^2 = 4\pi e^2\hat{n}_j/\gamma_b m$ is the square of the plasma frequency of beam component j, $\Omega_j^2 = (\hat{\omega}_j - \omega_j)(\omega_j + \hat{\omega}_j)$ is defined in Eq. (3), and $\Gamma_j(\omega)$ is defined by

$$\Gamma_j(\omega) = -1 + \left(\frac{\hat{\omega}_j - \omega_j}{2\hat{\omega}_j}\right)^l\sum_{n=4}^{l}\frac{l!}{n!(l-n)!}\frac{\omega - l\omega_j - k_z\beta_j c}{\omega - k_z\beta_j c + l\hat{\omega}_j - 2n\hat{\omega}_j}\left(\frac{\omega_j + \hat{\omega}_j}{\hat{\omega}_j - \omega_j}\right)^n. \quad (12)$$

Similarly, Eq. (6) can be expressed as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2}\right)\hat{A}(r) = -\sum_j\beta_j\hat{\psi}_j(r)\frac{\omega_{pj}^2}{\Omega_j^2 R_b}\Gamma_j(\omega)\delta(r - R_b), \quad (13)$$

where use has been again made of the approximation $p_z/\gamma_b m \simeq \beta_j c$ consistent with Eq. (1).

As the right-hand sides of the coupled differential equations (11) and (13) are equal to zero except at the surface of the beam $r = R_b$, they can be solved in a straightforward manner to give

$$C_k = \sum_j(1 - \beta_k\beta_j)(\omega_{pj}^2/2l\Omega_j^2)\Gamma_j(\omega)C_j, \quad k = e \text{ and } p. \quad (14)$$

In the case when the beams are located inside the cylindrical conducting wall with radius R_c, the term C_k in the left-hand side of Eq. (14) is replaced by $[1-(R_b/R_c)^2]^{-1}C_k$. Note that the absolute value of $\omega_{pj}^2\Gamma_j(\omega)/\Omega_j^2$ in Eq. (14) is of the order of unity or less. It follows from Eq. (14) that the condition for a nontrivial solution (C_j not all zero) is given by

$$1-(\omega_{pp}^2\omega_{pe}^2/l^2\Omega_p^2\Omega_e^2)\Gamma_p(\omega)\Gamma_e(\omega)=0, \qquad (15)$$

where use has been made of $\beta_p=-\beta_e\simeq 1$ and $\gamma_b^{-2}\ll 1$, which is consistent with present experimental parameters. Equation (15) when combined with Eq. (12), constitutes one of the main results of this Letter and can be used to investigate filamentation-stability properties for a broad range of system parameters.

In order to make the stability analysis tractable, we restrict the investigation of dispersion relation (15) to the case where both beams are in a cold-fluid rotational equilibrium characterized by $\omega_j\to\pm\hat{\omega}_j$. A careful examination of expression for $\Gamma_j(\omega)$ shows that[5]

$$\lim_{\omega_j\to\pm\hat{\omega}_j}\left[\frac{\omega_{pj}^2}{2l\Omega_j^2}\Gamma_j(\omega)\right]=\frac{\omega_{pj}^2}{2(\omega-k_z\beta_jc\mp l\omega_j)[\omega-k_z\beta_jc\mp(l-2)\omega_j]}. \qquad (16)$$

Moreover, we also assume that both beams are in a slow rotational equilibrium and have the same density, thereby giving $\omega_p=-\omega_e=\hat{\omega}_e=\hat{\omega}_p$ and $\hat{\omega}_e=\omega_{pe}=(4\pi\hat{n}_ee^2/\gamma_bm)^{1/2}$. Substituting Eq. (16) into Eq. (15) and defining $a=k_zc+l\omega_{pe}$, $b=k_zc+(l-2)\omega_{pe}$, we simplify the dispersion relation in Eq. (15) as $(\omega^2-a^2)(\omega^2-b^2)=\omega_{pe}^4$, which provides a necessary and sufficient condition $\omega_{pe}^4>a^2b^2=(k_zc+l\omega_{pe})^2(k_zc+l\omega_{pe}-2\omega_{pe})^2$ for instability. For the unstable branch, the perturbation is purely growing with the growth rate

$$\omega_i=\mathrm{Im}\,\omega=\left\{\left[\left(\frac{a^2-b^2}{2}\right)^2+\omega_{pe}^4\right]^{1/2}-\frac{a^2+b^2}{2}\right\}^{1/2}. \qquad (17)$$

The maximum growth rate of instability can occur at $a=0$ or $b=0$, thereby giving $(\omega_i)_m=(5^{1/2}-2)^{1/2}\omega_{pe}\simeq 0.5\omega_{pe}$, which is apparently independent of the azimuthal harmonic number l. The stability analysis of Eq. (15) for a broad range of rotational frequency ω_j is currently under investigation by the authors.

For colliding beams interacting over a finite distance $L_{\parallel}$, $k_z=2\pi m/L_{\parallel}$ ($m=1,2,\dots$) and the condition for $a=0$ becomes $L_{\parallel}(\omega_{pe}/c)=2\pi m/l$. The finite interaction length also imposes a severe condition for the instability to grow significantly before the beams exit; $\omega_iL_{\parallel}/c>1$. For parameters characteristic of recent colliding-beam experiments at DESY,[3] 8.5-GeV ($\gamma_b=17\,000$) electrons and positrons of 10^{12} particles with cross section 10^{-4} cm^2 and interaction length 2 cm, the density $n=5\times10^{15}/\mathrm{cm}^3$, and the growth rate is about $\omega_i\simeq 10^{10}$ sec^{-1}. Because the interaction region is limited to a length of $L=2$ cm, the above condition is marginally satisfied and the system is just over the instability threshold. Nonlinearly the beams become filamentated first; then the current filaments of the same sign attract each other to form a broader beam.

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