First-Order SU(3) Theorem Relating Weak and Electromagnetic Form Factors and Its Possible Experimental Verification

Alberto Sirlin
Department of Physics, New York Univeristy, New York, New York 10003
(Received 14 May 1979)

This paper discusses a first-order SU(3) theorem relating the weak and electromagnetic form factors of the pseudoscalar mesons with an error of second order in the symmetry breaking. Recent measurements of the K^0, K^-, and π^- mean square charge radii open the possibility of testing a prediction of this theorem in the near future.

In this short paper I discuss a theorem relating the weak and electromagnetic form factors of the pseudoscalar mesons, for a large range of momentum transfers, with a formal error of second order in the SU(3) breaking. Although, with a modification to be explained below, this result has been derived some time ago,[1] the possibility of its verification appeared very remote at that time and it has remained largely unknown. Recent measurements of the π^-, K^-, and K^0 mean square charge radii[2-5] indicate the possibility of testing an interesting relation implied by the theorem. On the theoretical side, the fact that theorems of this class are valid for arbitrary momentum transfers, modulo certain qualifications described later on, offers the possibility of studying the nature of SU(3) breaking in the matrix elements of current operators in novel and, perhaps, more probing ways.

The theorem states that

$$f_+^{(K^0\to\pi^-)}(t)=\tfrac{1}{2}[F^{(\pi^+)}(t)+F^{(K^+)}(t)]+F^{(K^0)}(t)-\tfrac{3}{2}[F^{(K)}(t)]_{us}+O(\lambda_3^2), \qquad (1)$$

where $f_+^{(K^0\to\pi^-)}(t)$ is the first-class form factor in the $K^0\to\pi^-$ amplitude; $F^{(a)}$ $(a=\pi^+, K^+, K^0)$ are the electromagnetic form factors of π^+, K^+, K^0; $[F^{(K)}(t)]_{us}$ is the contribution to $F^{(K)}(t)$ $(K=K^+$ or $K^0)$ of the unitary singlet component S_μ of the electromagnetic current; $t=q^2$ is the invariant momentum transfer; and λ_3 is the parameter that describes V-spin breaking.[6]

It is of course immaterial for physical applications whether we regard the error as being of second order in V-spin or SU(3) breakings, as these are of the same order of magnitude. In deriving this theorem I assume that SU(3) is broken by an operator λh_8 transforming as the eighth component of an octet, but otherwise I work in the limit of isospin invariance. Further, I take the electromagnetic current to be of the form $J_\mu^{\rm el}=J_\mu^3+(1/\sqrt{3})J_\mu^8+S_\mu$, where S_μ is a unitary singlet operator. In Ref. 1 it was assumed that S_μ does not exist and Eq. (1) was obtained without the unitary singlet contribution on the right-hand side. The structure of the modern gauge theories and the successful phenomenology of the new family of heavy mesons strongly indicate the existence of an S_μ component constructed from the heavy-quark fields: $S_\mu=\frac{2}{3}[\bar{c}\gamma_\mu c+\bar{t}\gamma_\mu t]-\frac{1}{3}\bar{b}\gamma_\mu b+\dots$. These unitary singlet contributions have the following properties:

(i) They vanish at $t=0$ because $\int S_0\, d^3x$ is a sum of terms proportional to the various heavy flavor number generators and their matrix elements between ordinary hadrons are zero.

(ii) They vanish in the SU(3) limit [for example, $\langle K^+|S_\mu|K^+\rangle=-\langle K^-|S_\mu|K^-\rangle$ by charge-conjugation invariance and $\langle K^+|S_\mu|K^+\rangle=\langle K^-|S_\mu|K^-\rangle$ in the SU(3) limit].

(iii) They are forbidden by Zweig's rule, the lowest-order quark diagrams involving the creation of heavy $q\bar{q}$ pairs which couple with the light quarks via three intermediate gluons. In this case, the quark loop diagrams are finite and vanish in the limit $m_c^2, m_t^2, \dots\to\infty$.

(iv) Current phenomenology indicates that S_μ has large matrix elements between the vacuum and the appropriate members of the new family of heavy mesons while there is no evidence of coupling to ordinary hadrons. If we assume that the imaginary parts of the unitary singlet form factors are dominated by the heavy mesons, then the values of the form factors at low momenta will be greatly suppressed by the large masses of the bosons and their very weak coupling to ordinary hadrons (see, for example, Ref. 12). These arguments suggest that the contributions of the unitary singlet term in Eq. (1) can be safely neglected, at least in the physical region of the weak decays.

Before discussing the implications of Eq. (1) I give, somewhat schematically, a more compact derivation than that developed in Ref. 1. The proof proceeds in two simple Lemmas:

Lemma 1.—We write $H(x)=H_0(x)+\lambda h_8(x)$

$=H_0(x)+\lambda_0 v_0(x)+\lambda_3 v_3(x)$, where H_0 is SU(3) symmetric and the symmetry-breaking term $\lambda h_8(x)$ is decomposed into V-spin singlet $(\lambda_0 v_0)$ and triplet $(\lambda_3 v_3)$ components. We may regard the V-spin symmetric part $H_0(x)+\lambda_0 v_0(x)$ as the unperturbed Hamiltonian density and study the perturbative expansion in powers of λ_3. In particular, consider the first-order response of the $K^0\to\pi^-$ form factors to the perturbation $\lambda_3 v_3$[7,8]:

$$\delta V_\mu^{(K^0\to\pi^-)}(p,p')=-i\lambda_3\lim_{\bar q\to q}\{\int d^4y e^{i\bar q\cdot y}\,{}_0\langle\pi^-(p')|T[J_\mu(y)v_3(0)]|K^0(p)\rangle_0-\delta M_\mu(p,p',\bar q)\},\tag{2}$$

where $J_\mu=J_\mu{}^4-iJ_\mu{}^5$, $q\equiv p-p'$ is the four-momentum transfer, $|a\rangle_0$ $(a=K^0,\pi^-)$ are the eigenstates in the limit of V-spin symmetry, and δM_μ subtracts in an appropriate manner the mass insertions of the perturbation in the external legs.[9] As explained in Refs. 7 and 8, Eq. (2) includes the corrections to the field renormalizations associated with the external legs with correct factors as well as the corrections induced on the zeroth-order form factors in going from the unperturbed to the corrected mass shells. As Eq. (2) is explicitly of order λ_3, we can work in the V-spin limit and consider, in particular, the symmetry transformation $G_V=C\exp(i\pi V_2)$, where $V_2=F_5$ is the second generator of V spin. Under G_V: $v_3\to-v_3$, $|K^0(p)\rangle_0\to|\pi^+(p)\rangle_0$, $|\pi^-(p')\rangle_0\to-|\bar K^0(p')\rangle_0$ while J_μ remains unchanged. Thus, the first term of Eq. (2) equals

$$\int_y e^{i\bar q\cdot y}\,{}_0\langle\bar K^0(p')|T[J_\mu(y)v_3(0)]|\pi^+(p)\rangle_0=\int_y e^{i\bar q\cdot y}\,{}_0\langle\pi^-(-p)|T[J_\mu(y)v_3(0)]|K^0(-p')\rangle_0,\tag{3}$$

where the last equality follows from the substitution law. After verifying that $\delta M_\mu(p,p',\bar q)$ satisfies[9] in our case the same symmetry relation [namely $\delta M_\mu(p,p',\bar q)=\delta M_\mu(-p',-p,\bar q)$], we obtain

$$\delta V_\mu^{(K^0\to\pi^-)}(p,p')=\delta V_\mu^{(K^0\to\pi^-)}(-p',-p).\tag{4}$$

If we write $\delta V_\mu^{(K^0\to\pi^-)}(p,p')\equiv\delta f_+(t)(p+p')_\mu+\delta f_-(t)(p-p')_\mu$, where $\delta f_\pm(t)$ represent the first-order corrections to the form factors, Eq. (4) implied

$$\delta f_+(t)=0\tag{5}$$

while it gives us no information about $\delta f_-(t)$.

Thus, we obtain the lemma that the physical form factor $f_+{}^{(K^0\to\pi^-)}(t)$ equals its V-spin limit with an error of second order in λ_3. Note that the momentum transfer is left invariant in Eq. (4) and is furthermore arbitrary so that these results are formally valid for any t [see, however, the observations below].

Lemma 2.—We consider the diagonal matrix elements of the conserved current $V_\mu{}^3=\frac{1}{2}[J_\mu{}^3+\sqrt{3}\,J_\mu{}^8]$ associated with the third generator of V spin:

$$\langle a(p')|V_\mu{}^3|a(p)\rangle\equiv f^{(a)}(t)(p+p')_\mu,\tag{6}$$

where $|a\rangle=|K^0\rangle,|\pi^-\rangle$ represent the exact physical states, and find

$$f_+{}^{(K^0\to\pi^-)}(t)=f^{(K^0)}(t)-f^{(\pi^-)}(t)+O(\lambda_3{}^2).\tag{7}$$

To prove Eq. (7) we first note that the corrections of first order in $\lambda_3 v_3$ to $f^{(\pi^-)}$ and $f^{(K^0)}$ are given by diagonal expressions analogous to Eq. (2) with $J_\mu\to V_\mu{}^3$. Under the symmetry operation $\exp(i\pi V_2)$ both $V_\mu{}^3$ v_3 change sign so that

$$ {}_0\langle\pi^-(p')|T[V_\mu{}^3(y)v_3(0)]|\pi^-(p)\rangle_0={}_0\langle K^0(p')|T[V_\mu{}^3(y)v_3(0)]|K^0(p)\rangle_0\tag{8}$$

which tells us that the corrections of order λ_3 cancel on the right-hand side of Eq. (7). Lemma 1 informs us that the same is true of the left-hand side. As the equality is true in the V-spin limit and the order-λ_3 corrections vanish, Eq. (7) follows. Finally, to establish contact with measurable quantities, we express the form factors $f^{(\pi^-)}(t)$ and $f^{(K^0)}(t)$ in terms of the electromagnetic form factors $F^{(a)}$ of π^+, K^+, and K^0 using only isospin relations valid to all orders in λh_8 and, therefore, to all orders in λ_0 and λ_3. When we remember the structure of $J_\mu{}^{\rm el}$, this leads to Eq. (1). The above derivation closely parallels that of Ref. 1 except that it gives the details in the proof of Lemma 2 and is directly based on the on-shell perturbative formulas of Refs. 7 and 8 rather than on the off-shell formulation of Ref. 1.

Although Eq. (1) is formally valid for arbitrary values of t, it cannot be applied in the resonance region. This, however, is easy to understand: A symmetry relation such as $[t-m_\rho{}^2]^{-1}=[t-m_{K^*}{}^2]^{-1}+O(\lambda)$ makes sense when t is spacelike or far away from the resonance region but it is certainly not meaningful when $t\sim m_\rho{}^2$. Perturbation expansions simply break down. Although the derivation of Eq. (1) gives us no detailed information about the nature of the $O(\lambda_3^2)$ terms, it is entirely possible that these symmetry relations are satisfied with good accuracy far from the resonance region, say in the spacelike region or near $t=0$.

Turning our attention to physical applications, we see that Eq. (1) implies $f_+^{(K^0\to\pi^-)}(0)=1+O(\lambda_3^2)$, the familiar result of the nonrenormalization theorem.[10,11] It is clear, however, that Eq. (1) contains considerable more information. In particular, if we differentiate Eq. (1) with respect to t and then set $t=0$ we obtain the prediction

$$6\lambda_+^{(K^0\to\pi^-)}/m_{\pi^+}^2=\tfrac{1}{2}[\langle r_{\pi^+}^2\rangle+\langle r_{K^+}^2\rangle]+\langle r_{K^0}^2\rangle+O(\lambda_3^2), \qquad (9)$$

where $\lambda_+^{(K^0\to\pi^-)}$ is the slope parameter in $K_L\to\pi^-+e^++\nu$ decay and $\langle r_a^2\rangle\equiv 6[dF^{(a)}/dt]_{t=0}$ (we use Feynman's metric) is the mean square charge radius of boson a. In obtaining Eq. (9) I have set $f_+^{(K^0\to\pi^-)}(0)=1$, consistent with our approximation, and have neglected the unitary singlet contributions.[12] The left-hand side equals[13] 0.360 $\pm$ 0.022 fm^2. With $\langle r_{\pi^+}^2\rangle=0.460\pm0.011$ fm^2 (Ref. 2), the preliminary result $\langle r_{K^+}^2\rangle=0.26\pm0.07$ fm^2 (Ref. 4), $\langle r_{K^0}^2\rangle=-0.054\pm0.026$ fm^2 (Ref. 5), the right-hand side equals 0.306 ± 0.044 fm^2, which is moderately encouraging. The comparison worsens if one uses the value $\langle r_\pi^2\rangle=0.31\pm0.04$ fm^2 quoted in Ref. 3. A more satisfactory test of the sum rule of Eq. (9) would require an improvement in the measurements of $\langle r_{K^+}^2\rangle$ and $\langle r_{K^0}^2\rangle$ and a solution to the apparent conflict in the measurements of $\langle r_\pi^2\rangle$ reported in Refs. 2 and 3.

Quite obviously, there are a number of theorems of the same class as Eq. (1) in the baryon octet sector. In Ref. 1 we already pointed out two interesting relations, the precise V-spin analog of Eq. (1). There are, however, more. These relations and their physical applications will be discussed in a separate communication.

The author is indebted to R. Brandt, A. García, J. Iliopoulos, J. Lowenstein, F. Neri, J. Sculli, S. S. Shei, and A. Wehmann for valuable conversations. This research was supported in part by the National Science Foundation, Grant No. PHY78-21503.

[1]A. Sirlin, Ann. Phys. (N.Y.) 61, 294 (1970). See especially Sect. 4.

[2]A. Quenzer *et al.*, Phys. Lett. 76B, 512 (1978).

[3]E. B. Dally *et al.*, Phys. Rev. Lett. 39, 1176 (1977).

[4]E. Tsyganov, in *Proceedings of the Nineteenth International Conference on High Energy Physics, Tokyo, Japan, August 1978*, edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Physical Society of Japan, Tokyo, 1979), p. 315.

[5]W. R. Molzon *et al.*, Phys. Rev. Lett. 41, 1213 (1978).

[6]The electromagnetic form factors are normalized so that $F_1^{(a)}(0)=1\ (-1)$ if particle a is positively (negatively) charged, and the weak form factor f_+ is defined so that at $t=0$ it equals 1 in the SU(3) limit. Thus,

$$\langle\pi^-(p')|(J^4-iJ^5)_\mu|K^0(p)\rangle = -[f_+^{(K^0\to\pi^-)}(t)(p+p')+f_-^{(K^0\to\pi^-)}(t)(p-p')]_\mu.$$

[7]Lowell S. Brown, Phys. Rev. 187, 2260 (1969).

[8]A. Sirlin, Rev. Mod. Phys. 50, 573 (1978). See especially Sect. III.

[9]The definition of $\delta M_\mu(p,p',\bar{q})$ can be inferred from Ref. 7 and from Eq. (3.8) of Ref. 8. In Eq. (2), the field operators evolve in time according to the symmetric Hamiltonian H_0.

[10]R. E. Behrends and A. Sirlin, Phys. Rev. Lett. 4, 186 (1960).

[11]M. Ademollo and R. Gatto, Phys. Rev. Lett. 13, 264 (1964).

[12]We may attempt to estimate the unitary singlet contributions to the right-hand side of Eq. (9) by a dispersive approach. If we assume that the imaginary part of the form factor is dominated by the J resonance, this contribution is $-\frac{3}{2}(f_{JK^+K^-}/f_J)(1/m_J^2)$, where $f_J^2/4\pi\approx 12$. Because of the largenss of m_J^2 and the very small partial decay rate $J\to K^++K^-$, this estimate gives a completely negligible result.

[13]C. Bricman *et al.*, Phys. Lett. 75B (1978).