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Generalized Gauge Hierarchies

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We generalize the analysis of coupling-constant renormalization in unified theories to allow for more than one scale of spontaneous symmetry breakdown between unification and $SU(3)\otimes SU(2)\otimes U(1)$. We find a relation between α, α_s, $\sin^2\theta_W$, and the proton lifetime which is independent of the details of the gauge hierarchy and holds in a large class of unified theories.

In an SU(5) unified theory of strong, weak, and electromagnetic interactions[1] the structure of the gauge hierarchy is unique. If this theory is to describe physics, the SU(5) symmetry must be broken at a very large momentum, $\sim 10^{15}$ GeV, down to $SU(3)\otimes SU(2)\otimes U(1)$.[2] This symmetry is further broken down to $SU(3)\otimes U(1)$ at $M_W \sim 80$ GeV. There is no choice for the first symmetry breakdown because the only proper subgroup of SU(5) which contains $SU(3)\otimes SU(2)\otimes U(1)$ is $SU(3)\otimes SU(2)\otimes U(1)$ itself. Consequently, the effect of the gauge hierarchy on the renormalization of $\sin^2\theta_W$ is unique, and any reasonable choice for the color SU(3) coupling α_s implies $\sin^2\theta_W \simeq 0.2$. Experimentally, $\sin^2\theta_W = 0.23 \pm 0.01$.[3] This is closer to 0.2 than it is to $\frac{3}{8}$ (the naive unification value) which is nice, but it seems to be significantly different from the SU(5) prediction. What is going on? One possibility is that the unifying group may be some larger group which contains SU(5). Then, the gauge symmetry may break down to $SU(3)\otimes SU(2)\otimes U(1)$ in several steps. In this paper, we generalize the analysis of Ref. 2 to this more general situation. We find a useful formula with a simple physical interpretation. We then apply it in some interesting examples.

If a simple gauge group G is broken down to $SU(3)\otimes SU(2)\otimes U(1)$ in N steps, the gauge hierarchy of the theory can be characterized by $N+1$ masses, μ_x, $x=0$ to N, describing the strengths of the various steps of symmetry breaking. We assume that $\mu_{x-1} \ll \mu_x$. There is a large region between μ_{x-1} and μ_x where the physics can be described by an effective field theory with an S^x gauge symmetry. S^x is the subgroup of G which is left unbroken in steps x through N of the hierarchy.[4] S^x is broken down to S^{x-1} at a mass μ_{x-1}. The gauge bosons associated with S^x but not with S^{x-1} get mass of the order of μ_{x-1}.

The subgroup S^x is a product of simple factors and U(1) factors

$$S^x = \prod_\alpha s_\alpha^{\,x}, \tag{1}$$

where $s_\alpha^{\,x}$ is either a simple non-Abelian subgroup or a U(1). S^1, for example, is $U(1)\otimes SU(2)\otimes SU(3)$; so we can take $s_1^{\,1} = U(1)$, $s_2^{\,1} = SU(2)$, and $s_3^{\,1} = SU(3)$. S^0 is $SU(3)\otimes U(1)$ and $\mu_0 \sim M_W$. Denote the generators of $s_\alpha^{\,x}$ by $T_{\alpha i}^{\,x}$. The unification provides a natural normalization of the generators. Choose a convenient representation of G (for example, the adjoint representation) so that

$$\mathrm{tr}(T_{\alpha i}^{\,x} T_{\beta j}^{\,x}) = \lambda \delta_{\alpha\beta}\delta_{ij}, \tag{2}$$

where λ is any convenient constant.

We will choose λ so that the electric charge Q is

$$Q = T_{23}^{\,1} + (\tfrac{5}{3})^{1/2} T_{11}^{\,1} \tag{3}$$

in terms of the generators of S^1 [$T_{23}^{\,1}$ is the third component of weak SU(2) and $T_{11}^{\,1}$ is the U(1) generator]. The factor $(\frac{5}{3})^{1/2}$ in Eq. (3) is characteristic of groups G with the $SU(3)\otimes SU(2)\otimes U(1)$ embedded in an SU(5) subgroup [in Ref. 2, $(\frac{5}{3})^{1/2} = -C$].

If S^x contains more than one U(1) factor, we choose all but (at most) one to be orthogonal to the electric charge [$\mathrm{tr}(QT_{\alpha 1}^{\,x}) = 0$]. The orthogonal U(1) generators do not get involved in the physics of S^1, so we will ignore them and assume that S^x contains at most one U(1) subgroup.

Because of the nested gauge subgroup structure, we can express the generators of S^x as linear combinations of some subset of the generators of S^y for $y>x$, as

$$T_{\alpha i}{}^{x}=\sum_{\beta j}C_{\alpha i\beta j}{}^{x,y}T_{\beta j}{}^{y}. \quad (4)$$

The gauge invariance [plus the restriction to a single U(1) subgroup for each x] implies

$$\sum_{k}C_{\alpha i\gamma k}{}^{x,y}C_{\beta j\gamma k}{}^{x,y}=\delta_{\alpha\beta}\delta_{ij}P_{\alpha\gamma}{}^{x,y}, \quad (5)$$

where

$$P_{\alpha\beta}{}^{x,y}=\sum_{j}|C_{\alpha i\beta j}{}^{x,y}|^2. \quad (6)$$

$P_{\alpha\beta}{}^{x,y}$ is the probability that the $s_\alpha{}^x$ subgroup of S^x exists in the $s_\beta{}^y$ subgroup of S^y. The P's satisfy

$$P_{\alpha\beta}{}^{x,x}=\delta_{\alpha\beta}, \quad (7a)$$

$$\sum_{\beta}P_{\alpha\beta}{}^{x,y}=1, \quad (7b)$$

$$P_{\alpha\beta}{}^{x,y}=\sum_{\gamma}P_{\alpha\gamma}{}^{x,z}P_{\gamma\beta}{}^{z,y}, \quad x\leq z\leq y. \quad (7c)$$

Equation (7a) follows from $C_{\alpha i\beta j}{}^{x,x}=\delta_{\alpha\beta}\delta_{ij}$ and Eq. (6). Equation 7(b) follows from Eqs. (2), (4), and (6). Equation (7c) follows from Eq. (5) and the multiplication law for the C's,

$$\sum_{\gamma k}C_{\alpha i\gamma k}{}^{x,z}C_{\gamma k\beta j}{}^{z,y}=C_{\alpha i\beta j}{}^{x,y}. \quad (8)$$

We now state and prove our main result. If $g_\alpha{}^x(E)$ are the gauge coupling constants for the $s_\alpha{}^x$ subgroups renormalized at E ($\mu_{x-1}\leq E\leq\mu_x$), then to second order in g and in the approximation of Ref. 2, $\mu_{x-1}\ll\mu_x$ for all x and sharp transitions between different regions, they satisfy

$$\frac{1}{g_\alpha{}^x(\mu_{x-1})^2}=\frac{1}{g(\mu_N)^2}+\sum_{y=x}^{N}\ln\frac{\mu_y}{\mu_{y-1}}\sum_{\beta}P_{\alpha\beta}{}^{x,y}2b_\beta{}^y. \quad (9)$$

g is the gauge coupling constant of the unifying group G. $b_\beta{}^y$ is the constant which appears in the β function for $g_\alpha{}^x$,

$$\beta_\alpha{}^x(g_\alpha{}^x)=b_\alpha{}^x g_\alpha{}^{x\,3}+O(g_\alpha{}^{x\,5}). \quad (10)$$

Equation (9) is the obvious generalization of the result of Ref. 2 to the generalized gauge hierarchy. The result can be stated as follows: In the yth region, the subgroup $s_\alpha{}^x$ exists in $s_\beta{}^y$ with probability $P_{\alpha\beta}{}^{x,y}$; so the renormalization of its coupling constant $g_\alpha{}^x$ is governed by the average of the $b_\beta{}^y$'s weighted with these probabilities.

To prove Eq. (9), we first show that the couplings $g_\alpha{}^x(\mu_x)$ and $g_\beta{}^{x+1}(\mu_x)$ are related as follows:

$$\frac{1}{g_\alpha{}^x(\mu_x)^2}=\sum_{\beta}P_{\alpha\beta}{}^{x,x+1}\frac{1}{g_\beta{}^{x+1}(\mu_x)^2}. \quad (11)$$

The $W_i{}^y$ associated with s^y couple to $g_\alpha{}^y T_{\alpha i}{}^y$. Consistency with Eq. (4) requires

$$W_i{}^x=g^x\sum_{\beta j}C_{\alpha i\beta j}{}^{x,x+1}W_{\beta j}{}^{x+1}/g_\beta{}^{x+1}. \quad (12)$$

The normalization condition for the W fields plus Eq. (6) yields Eq. (11).

We now prove Eq. (9) by induction. For $x=N$, Eq. (9) is correct. In fact, it is just the result of Ref. 2 for a single-step hierarchy. Suppose it is true for $x=z+1$. Standard renormalization-group techniques give

$$\frac{1}{g_\alpha{}^z(\mu_{z-1})^2}=\frac{1}{g_\alpha{}^z(\mu_z)^2}+2b_\alpha{}^z\ln\frac{\mu_z}{\mu_{z-1}}. \quad (13)$$

Equation (11) can be used on the first term on the right-hand side yielding

$$\frac{1}{g_\alpha{}^z(\mu_z)^2}=\sum_{\beta}P_{\alpha\beta}{}^{z,z+1}\frac{1}{g_\alpha{}^{z+1}(\mu_z)^2}. \quad (14)$$

By assumption, this is

$$\sum_{\beta}P_{\alpha\beta}{}^{z,z+1}\frac{1}{g(\mu_N)^2}+\sum_{\beta}P_{\alpha\beta}{}^{z,z+1}\times\sum_{y=z+1}^{N}\ln\frac{\mu_y}{\mu_{y-1}}\sum_{y}P_{\beta\gamma}{}^{z+1,y}2b_\gamma{}^y. \quad (15)$$

Using Eq. (7), we can write Eq. (15) as

$$\frac{1}{g(\mu_N)^2}+\sum_{y=z+1}^{N}\ln\frac{\mu_y}{\mu_{y-1}}\sum_{\beta}P_{\alpha\beta}{}^{z,y}2b_\beta{}^y. \quad (16)$$

Combining Eq. (16) with the second term on the right-hand side of Eq. (13) establishes Eq. (9) for $x=z$ and completes the proof.

As a demonstration of the power of Eq. (9), we will consider an especially simple but interesting class of generalizations of SU(5). Suppose that G is SU(L) with SU(5) embedded in such a way that the L-dimensional representation consists of an SU(5) 5 and $L-5$ neutral singlets. We will show that with reasonable assumptions we can derive a relation between α, α_s, $\sin^2\theta_W$, and the proton lifetime which is independent of the form of the gauge hierarchy.

Define $\mu_N=M$ to be the mass characterizing the strongest symmetry-breaking step which separates the SU(3) and SU(2) subgroups of the SU(5) subgroup of SU(L) into different $s_\alpha{}^N$. There may be yet stronger breakdowns, but they are irrelevant because the SU(5) is still unified. The vector bosons which mediate quark-number–nonconserving interactions have mass of order M. M is the analog of the unifying mass in SU(5).

The subgroup S^x for $x \leq N$ must consist of an $SU(m_x)$ which contains the weak SU(2), an $SU(n_x)$ which contains color SU(3) and a $U(1)^x$ which contains the rest of the charge. There may be other subgroups but they are completely neutral and irrelevant to α, α_s, and $\sin^2\theta_W$, and so we will ignore them. Thus we can take

$$\begin{aligned} s_1{}^x &= U(1)^x , \\ s_2{}^x &= SU(m_x) \supset SU(2) , \\ s_3{}^x &= SU(n_x) \supset SU(3) . \end{aligned} \tag{17}$$

The form of the $T_{\alpha i}{}^x$ generators which contribute to low-energy physics is now fixed by the nature of the SU(5) embedding and the normalization condition, Eqs. (2) and (3). We can compute the P's. We find

$$\begin{aligned} P_{22}{}^{1,x} &= P_{33}{}^{1,x} = 1 , \\ P_{12}{}^{1,x} &= \tfrac{3}{5}(m_x - 2)/m_x , \\ P_{13}{}^{1,x} &= \tfrac{2}{5}(n_x - 3)/n_x , \\ P_{11}{}^{1,x} &= \tfrac{6}{5}(m_x + n_x)/m_x n_x . \end{aligned} \tag{18}$$

We can now calculate $g_\alpha{}^1(\mu_0)$, $\alpha = 1$ to 3:

$$\frac{1}{g_\alpha{}^1(\mu_0)^2} = \frac{1}{g(\mu_N)^2} + \sum_{y=x}^{N} \ln\left(\frac{\mu_y}{\mu_{y-1}}\right) \sum_{\beta=1}^{3} P_{\alpha\beta}{}^{1,y} 2 b_\beta{}^y , \tag{19}$$

where

$$\begin{aligned} 48\pi^2 b_1{}^y &= 2F_1{}^y , \\ 48\pi^2 b_2{}^y &= -11 m_x + 2F_2{}^y , \\ 48\pi^2 b_3{}^y &= -11 n_x + 2F_3{}^y . \end{aligned} \tag{20}$$

The constants $F_a{}^y$ depend on the number of spin-$\frac{1}{2}$ and scalar particles with mass less than μ_y. To obtain a simple result, we must assume that $F_\alpha{}^y$ is independent of α. This will be the case if within each representation of the SU(5) subgroup all particles have masses of the same order of magnitude.

Assuming $F_1{}^y = F_2{}^y = F_3{}^y$, we can find a combination of coupling constants for which all y dependence disappears from $\sum_\beta P_{\alpha\beta} b_\beta$ in Eq. (19). It is

$$\frac{1}{g_1{}^1(\mu_0)^2} - \frac{3}{5}\frac{1}{g_2{}^1(\mu_0)^2} - \frac{2}{5}\frac{1}{g_3{}^1(\mu_0)^2} . \tag{21}$$

But

$$\begin{aligned} g_3{}^1(\mu_0)^2/4\pi &= \alpha_s(\mu_0) , \\ g_2{}^1(\mu_0)^2/4\pi &= \alpha(\mu_0)/\sin^2\theta_W , \\ g_1{}^1(\mu_0)/4\pi &= \tfrac{5}{3}\alpha(\mu_0)/\cos^2\theta_W , \end{aligned} \tag{22}$$

and $\mu_0 = M_W$, and so we get the following relation

$$\frac{\cos 2\theta_W}{\alpha(M_W)} - \frac{2}{3}\frac{1}{\alpha_s(M_W)} \simeq \frac{22}{3\pi} \ln \frac{M}{M_W} . \tag{23}$$

For $\sin^2\theta_W \simeq 0.23$ and $\alpha_s(M_W) \simeq 0.15$, $\alpha(M_W) \simeq 1/128$.[5] This gives $M \simeq 10^{14}$ GeV, corresponding to a proton lifetime $\sim 10^{29}$ yr.

It is important to note that the above analysis applies even if some of the stages of symmetry breaking are dynamical. Equation (23) will be satisfied in SU(N) "technicolor" theories[6] if the embedding of SU(5) is such that the N is a 5 plus $N - 5$ singlets. If the embedding of SU(5) in G is more complicated, one can use Eq. (9) to obtain bounds on the coupling constants and the unifying mass. However, it is more informative for small groups like O(10) and E(6) to analyze each of the possible gauge hierarchies separately.[7] The example of O(10) is discussed in detail in Ref. 7.

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