Random Magnetic Fields, Supersymmetry, and Negative Dimensions

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We prove the equivalence, near the critical point, of a D-dimensional spin system in a random external magnetic field with a $(D-2)$-dimensional spin system in the absence of a magnetic field. This is due to the hidden supersymmetry of the associated stochastic differential equation. We identify a space with one anticommuting coordinate with a space having negative dimensions -2.

The critical behavior of a spin system in a random external magnetic field (i.e., the infrared behavior of a scalar field theory in presence of a random external source) has recently been investigated.[1] By explicit computations it has been found[1,2] that the values of the most-infrared-divergent diagrams[3] in dimensions D are equal to the values of the same diagrams without magnetic fields in dimensions $D-2$. In this Letter we show that this apparently mysterious result has a simple geometrical interpretation, which stems from a hidden supersymmetry[4] of the associated stochastic equation.

Let us define the free energy F_R averaged over a Gaussian random magnetic field by the functional integrals[5]:

$$F[\boldsymbol{h}]=\ln \int \mathfrak{D}_{\varphi} \exp\{-\int d^Dx[\mathfrak{L}(x)+\boldsymbol{h}(x)\varphi(x)]\},$$
$$F_R=\int \mathfrak{D}h F[\boldsymbol{h}]\exp[-\tfrac{1}{2}\int d^Dx\, h^2(x)], \qquad (1)$$
$$\mathfrak{L}(x)=-\tfrac{1}{2}\varphi(x)\Delta\varphi(x)+V(\varphi(x)).$$

For definiteness we can consider $V(\varphi)=\frac{1}{2}m^2\varphi^2+g\varphi^4$. The perturbative expansion in g for F_R can be easily constructed either by direct inspection, or by using the replica trick.[6] The most-infrared-divergent diagrams contain the maximum number of h^2 insertions, as follows from dimensional analysis. If the other diagrams are neglected we obtain the tree approximation for $F(h)$. Using the correspondence between the tree approximation[7] and the classical nonlinear differential equation, we find that in this limit the two-point Green's function is given by

$$\langle \varphi(x)\varphi(0)\rangle_R \sim \int \mathfrak{D}h\, \varphi_h(x)\varphi_h(0)\exp[-\tfrac{1}{2}\int d^Dy h^2(y)], \qquad (2)$$

where $\varphi_h(x)$ is the solution of the equation

$$-\Delta\varphi+V'(\varphi)+h=0. \qquad (3)$$

Equation (3) can also be regarded as a differential stochastic equation, h being a stochastic Gaussian function having autocorrelation $\langle h(x)h(y)\rangle=\delta^D(x-y)$. The results of Refs. 1 and 2 imply that the Green's functions of the stochastic differential equations (3) are the same as those generated by the Lagrangian of Eq. (1) in $D-2$ dimensions. Let us see why.

Using standard manipulations,[8] we find

$$\langle\varphi(x)\varphi(0)\rangle \sim \int \mathfrak{D}\varphi\mathfrak{D}h\,\varphi(x)\varphi(0)\,\delta(-\Delta\varphi+V'(\varphi)+h)\det[-\Delta+V''(\varphi)]\exp[-\tfrac{1}{2}\int h^2(y)d^Dy]$$
$$\sim \int \mathfrak{D}\varphi\,\mathfrak{D}\omega\mathfrak{D}\psi\exp[-\int d^Dy\,\mathfrak{L}_R(y)]\,\varphi(x)\varphi(0), \qquad (4)$$
$$\mathfrak{L}_R=-\tfrac{1}{2}\omega^2+\omega[-\Delta\varphi+V'(\varphi)]+\bar\psi[-\Delta+V''(\varphi)]\psi,$$

where ψ is an anticommuting scalar field[9] (a ghost field). The Lagrangian $\mathfrak{L}_R$ is invariant under the supersymmetry transformations:

$$\delta\varphi=-\bar a\epsilon_\mu x_\mu\psi, \quad \delta\omega=2\bar a\epsilon_\mu\partial_\mu\psi,$$
$$\delta\psi=0, \quad \delta\bar\psi=\bar a(\epsilon_\mu x_\mu\omega+2\epsilon_\mu\partial_\mu\varphi), \qquad (5)$$

$\bar a$ being an infinitesimal anticommuting number and ϵ_μ an arbitrary vector. The invariance under these supersymmetry transformations [Eq. (5)] is quite unexpected.[10] It is useful to introduce the superspace[4] characterized by a D-dimensional commuting coordinate x and by an anticommuting coordinate θ ($\theta^2=\bar\theta^2=\theta\bar\theta+\bar\theta\theta=0$) and the superfield,

$$\Phi(x,\theta)=\varphi(x)+\bar\theta\psi(x)+\bar\psi(x)\theta+\theta\bar\theta\omega(x). \qquad (6)$$

Higher orders in θ are identically zero as a result of the anticommuting properties of θ. The action [Eq. (4)] can be written as $\int d^D x d\theta \mathcal{L}_{ss}(\Phi)$, with

$$\mathcal{L}_{ss}(\Phi) = -\tfrac{1}{2}\Phi\Delta_{ss}\Phi + V(\Phi), \tag{7}$$

where $\Delta_{ss} = \Delta + \partial^2/\partial\bar\theta\partial\theta$ is the Laplacian in the superspace and the integration in θ selects the term proportional to $\theta\bar\theta$ [e.g., $\int d\theta\, \Phi(x,\theta) = -\pi^{-1}\omega(x)$].[11]

The supersymmetry transformations [Eq. (5)] are simply rotations in superspace leaving invariant the metric $x^2 + \theta\bar\theta$. We argue that the superspace (x,θ) is equivalent to an ordinary $(D-2)$-dimensional space. Indeed a space with only one anticommuting coordinate θ is equivalent to an ordinary space with negative dimensions -2. (Space with negative dimensions are defined by analytic continuations[12] from positive dimensions.) This can be seen from the relation

$$\int d\theta f(\bar\theta\theta) = -\frac{1}{\pi}\frac{d}{dz}f(z)\Big|_{z=0} = \lim_{D\to -2}\int d^D r f(r^2) = \lim_{D\to -2} S_D \int r^{D-1} dr f(z^2), \quad S_D = 2\pi^{D/2}/\Gamma(\tfrac{1}{2}D). \tag{8}$$

S_D is the surface of the unit sphere in D dimensions.

Let us consider a space of dimension $D-2$ and formally decompose it as the sum of a space of dimension D and of another space of dimension -2. The previous argument implies that an ordinary space of dimension $D-2$ is equivalent to the D-dimensional superspace. The precise meaning of the equivalence is the following:

$$\int d^{D-2}x F(Y_i x, x^2) = \int d^D x d\theta\, F(Y_i x, x^2 + \bar\theta\theta), \tag{9}$$

where Y_i are some $(D-2)$-dimensional vectors. For example,

$$\int d^{D-2}x f(x^2) = \int d^D x d\theta f(x^2 + \bar\theta\theta) = -\pi^{-1}\int d^D x f'(x^2) = \int d^{D-2}x f(x^2). \tag{10}$$

Equation (9) is sufficient to prove, at all orders in perturbation theory, that the Green's functions computed in the $D-2$ space are the same as those computed in the D-dimensional superspace. Indeed, the perturbative expansion for the Lagrangian (7) [which is equivalent to the stochastic Eq. (3)] can be written directly from Feynmann's rule in configuration superspace using the technique of superpropagator.[13] The final integrals have the form of Eq. (9) and the equivalence of the stochastic Eq. (3) with the $(D-2)$-dimensional field theory is therefore proved in the perturbative expansion. It has its root in the hidden supersymmetry of the system and in the geometrical equivalence of an anticommuting-variable space with a negative-dimensional space.

It may be useful to establish this equivalence rigorously beyond perturbation theory. The stochastic differential Eq. (3) may provide us with a different framework to study the properties of a field theory. It would be quite interesting to see if and how this formalism can be extended to gauge theories.

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