Variational Approach to Strong-Coupling Quantum Chromodynamics

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We consider a superposition of string configurations as a meson trial wave function in quantum chromodynamics and regard its amplitude as a variational parameter. By minimizing the meson energy we obtain a Schrödinger equation and an effective Hamiltonian for the amplitude.

At present, quantum chromodynamics (QCD) appears to be the most promising candidate theory of the strong interactions.[1] It offers an attractive possibility of achieving quark confinement at the same time with asymptotic freedom.

So far, QCD has been extensively studied in its two extreme regimes: weak-coupling regime and strong-coupling regime. The weak-coupling expansion is, of course, suited to the study of the relativistic high-energy behavior of the theory based on asymptotic freedom and is also used to describe vacuum tunneling phenomena. On the other hand, the strong-coupling dynamics has to do with the static long-distance properties of QCD.

So far as the confining aspect of the theory is concerned, a beautiful simplicity has been observed in the lattice formulation of strong-coupling theory.[2,3] Here, unlike the weak-coupling treatment, we can deal only with gauge-invariant excitations, mesons and baryons, and can set up a simple perturbation scheme to compute the hadron spectrum. It is also widely known that in this formulation we encounter stringlike objects composed of electric flux lines reminiscent of those in the string model of hadrons. Up to now lower-order perturbation-theory calculations have been done and some guesses made on the behavior of the theory in the continuum limit.

In this paper we want to propose a new approach to strong-coupling QCD based on a variational technique. We consider a superposition of strings in various configurations as a meson trial wave function in QCD and regard its amplitude, a functional of the string, as a variational parameter. By minimizing the meson energy we obtain an effective Hamiltonian and a Schrödinger equation for the amplitude.

Our effective Hamiltonian acting on the string functional is written in terms of string variables defined in one spatial dimension. Thus our variational technique amounts to a transformation of the original gauge-field variables in three space dimensions to one-dimensional string variables. By this transformation the kinetic and potential terms in the original Hamiltonian are mapped into potential and kinetic terms in the effective Hamiltonian and hence this is a kind of duality transformation.[4]

Let us start by defining notations in this paper. In order to avoid ultraviolet problems we work with the Hamiltonian formulation of SU(2) lattice gauge theory[3] where

$$E_i^a(\vec{x}), \quad a, i = 1, 2, 3,$$

$$U_{1/2}(\Gamma)\exp[(i/\hbar)\int_{\vec{x}_1}^{\vec{x}_2}\vec{A}_i(\vec{x})\cdot\tfrac{1}{2}\vec{\tau}d^3x_i],$$

$$H = \sum_{\vec{x}}\frac{g^2}{2a}E^2(\vec{x}) - \sum_{\gamma}\frac{2}{ag^2}\operatorname{Tr}[U_{1/2}(\gamma)],$$

are the electric field, the fundamental phase factor associated with a path Γ starting from $\vec{x}_1$ and ending at $\vec{x}_2$, and the Hamiltonian, respectively. $\operatorname{Tr}U(\gamma)$ is the trace of a group element $U(\gamma)$ for a plaquette γ and it is real for SU(2); a is the lattice constant; $U(\Gamma)$ is the creation operator of an electric flux line when we define a vacuum annihilated by the electric field,

$$E_i^a(\vec{x})|0\rangle = 0. \qquad (1)$$

In fact, by using a commutation relation we obtain

$$\vec{E}_t^a(\vec{x}_1)U_{1/2}(\Gamma)|0\rangle = \tfrac{1}{2}\tau^a U_{1/2}(\Gamma)|0\rangle, \qquad (2)$$

where t means the tangential direction to the string at $\vec{x}_1$. Then by introducing quark fields we construct a state

$$q^\dagger(\vec{x}_1)U(\Gamma)q(\vec{x}_2)|0\rangle,$$

which is interpreted as a gauge-invariant meson state with an electric flux line at Γ. In this paper we only consider infinitely massive quarks and treat them as external sources.

We now define our meson trial wave function to be a linear superposition of these states. Denoting the amplitude of the occurrence of the string configuration Γ by a functional $f[\Gamma]$, it is given by[5]

$$|M\rangle=\sum_\Gamma \tfrac{1}{2}\sqrt{2}\, f[\Gamma] q^\dagger(\vec{x}_1) U_{1/2}[\Gamma] q(\vec{x}_2)|0\rangle. \tag{3}$$

In Eq. (3) we restrict ourselves to strings with $I=\frac{1}{2}$ representation and do not consider possible internal excitations of the string corresponding to higher representations. Also we restrict the sum over Γ to those paths which are not self-intersecting and have no disconnected loops. We regard $f[\Gamma]$ to be our variational functional. The normalization of the meson wave function is computed by summing over the internal states of quarks,

$$\langle M|M\rangle=\sum_\Gamma\sum_{\Gamma'}\tfrac{1}{2}f^*[\Gamma']f(\Gamma)\langle 0|\operatorname{Tr}U^\dagger(\Gamma')U(\Gamma)|0\rangle=\sum_\Gamma|f(\Gamma)|^2=1, \tag{4}$$

where we have used

$$\langle 0|\operatorname{Tr}U^\dagger(\Gamma')U(\Gamma)|0\rangle=2\delta_{\Gamma'\Gamma}. \tag{5}$$

Now let us compute the expectation value of the Hamiltonian in our meson trial wave function. The effect of the electric field is easy to compute. Using Eq. (2) we obtain

$$\langle M|\sum\vec{E}_i^{\,2}(\vec{x})|M\rangle=(C_2/a)\sum_\Gamma l[\Gamma]|f[\Gamma]|^2. \tag{6}$$

Here $C_2=\frac{3}{4}$ is the Casimir operator of SU(2) and $l[\Gamma]$ is the length of the string Γ. Thus the expection value of the electric field term in the Hamiltonian is proportional to the average length of the string in the meson state.

The expectation value of the magnetic field term,

$$\langle M|\sum_\gamma \operatorname{Tr}U(\gamma)|M\rangle=\tfrac{1}{2}\sum_\Gamma\sum_{\Gamma'}\sum_\gamma f^*[\Gamma']f[\Gamma]\langle 0|\operatorname{Tr}\{U^\dagger[\Gamma'][\operatorname{Tr}U(\gamma)]U[\Gamma]\}|0\rangle, \tag{7}$$

is computed as follows: First we notice that every plaquette γ which does not overlap with Γ gives a vanishing contribution,

$$\sum_{\gamma\cap\Gamma=0}\langle 0|\operatorname{Tr}\{U^\dagger[\Gamma'][\operatorname{Tr}U(\gamma)]U[\Gamma]\}|0\rangle=0, \tag{8}$$

because of Eq. (5). Therefore we need to consider only those plaquettes in Eq. (7) which overlap with Γ. Then consider, for example, a situation where a plaquette $\gamma=\gamma_1\gamma_2\gamma_3\gamma_4$ has a common link, say, γ_1 with $\Gamma=\Gamma_1\gamma_1\Gamma_2$. Then by the addition of isospin the link γ_1 carries either $I=0$ or $I=1$ representation,

$$[\operatorname{Tr}U_{1/2}(\gamma)]U_{1/2}(\Gamma)=\tfrac{1}{2}U_{1/2}(\Gamma_1\gamma_2\gamma_3\gamma_4\Gamma_2)+\tfrac{1}{2}U_{1/2}(\Gamma_1\tfrac{1}{2}\tau_\alpha\gamma_2\gamma_3\gamma_4\tfrac{1}{2}\tau_\beta\Gamma_2)[U_1(\gamma_1)]_{\alpha\beta}. \tag{9}$$

The $I=0$ piece corresponds to a deformation of the string Γ to $\Gamma_1\gamma_2\gamma_3\gamma_4\Gamma_2$ and the $I=1$ piece corresponds to an internal excitation of the string. The $I=1$ piece, however, is orthogonal to our trial wave function $|M\rangle$ consisting only of $I=\frac{1}{2}$ representation and hence does not contribute to the matrix element.

In general, the action of a plaquette γ is written as

$$[\operatorname{Tr}U_{1/2}(\gamma)]U_{1/2}[\Gamma]=\tfrac{1}{2}U_{1/2}[\Gamma U\gamma]+\tfrac{1}{2}(I=1\text{ piece}), \tag{9a}$$

where $\Gamma U\gamma$ is defined not to include overlapping link(s).

Now we define an operator $O(\gamma)$ by

$$O(\gamma)g[\Gamma]=g[\Gamma U\gamma] \tag{10}$$

for any functional g of Γ. $O(\gamma)$ is a unitary operator in the space of functionals. Then we obtain

$$\begin{aligned}\langle M|\sum_\gamma\operatorname{Tr}U(\gamma)|M\rangle&=\tfrac{1}{2}\langle M|\sum_\gamma[\operatorname{Tr}U(\gamma)+\operatorname{Tr}U^\dagger(\gamma)]|M\rangle\\&=\tfrac{1}{4}\sum_{\Gamma'}\sum_\Gamma\sum_\gamma{}'\{f^*[\Gamma']f[\Gamma]\langle 0|\operatorname{Tr}U^\dagger[\Gamma']\operatorname{Tr}U[\Gamma U\gamma]+\operatorname{Tr}U[\Gamma' U\gamma]\operatorname{Tr}U[\Gamma]|0\rangle\}\\&=\tfrac{1}{4}\sum_\Gamma\sum_\gamma{}'\{(Of)^*[\Gamma]f[\Gamma]+f^*[\Gamma](Of)[\Gamma]\}\\&=\tfrac{1}{4}\sum_\Gamma\sum_\gamma{}'\{f^*[\Gamma][O(\Gamma)+O^\dagger(\gamma)]f[\Gamma]\}.\end{aligned} \tag{11}$$

$\sum'$ means the summation over overlapping γ's. Collecting together formulas (6), (8), and (11), we find that the expectation value of the Hamiltonian is given by

$$E=\langle M|H|M\rangle=\frac{C_2g^2}{a^2}\sum_\Gamma l[\Gamma]f[\Gamma]^2-\frac{1}{ag^2}\sum_\Gamma\sum_\gamma{}'f^*[\Gamma]\tfrac{1}{2}[O(\gamma)+O^\dagger(\gamma)]f[\Gamma]. \tag{12}$$

Then by minimizing the meson energy by varying $f[\Gamma]$ under the normalization condition Eq. (4), we obtain the effective Hamiltonian for the string functional:

$$H_{\rm eff} f[\Gamma]=\{(C_2 g^2/2a^2)l[\Gamma]-(1/ag^2)\textstyle\sum_\gamma{}' \frac{1}{2}[O(\gamma)+O^\dagger(\gamma)]\}f[\Gamma]=Ef[\Gamma]. \tag{13}$$

This is the basic result of this paper.

Now in order to discuss the physical content of this Hamiltonian, it is convenient to introduce an approximation by replacing a zigzagging line Γ by a smooth curve $x(s)$ and to rewrite our effective Hamiltonian taking the naive continuum limit; the parameter s is the proper distance along the string. Let us first introduce an orthonormal set of functions $\{\varphi_n(\sigma)\}$ on the string and expand the string coordinate

$$\vec{x}(\sigma)=\textstyle\sum_n \varphi_n(\sigma)\vec{x}_n, \tag{14}$$

here σ is an arbitrary parametrization of the path. Next we define the functional derivative[6]

$$\frac{\delta}{\delta\vec{x}(\sigma)}=\sum_n \varphi_n(\sigma)\frac{\partial}{\partial\vec{x}_n}, \quad \frac{\delta}{\delta\vec{x}(s)}=\frac{d\sigma}{ds}\frac{\delta}{\delta\vec{x}(\sigma)}, \tag{15}$$

which has the commutation relation

$$\left[x_i(\sigma), \frac{\delta}{\delta x_j(\sigma')}\right]=-\delta_{ij}\delta(\sigma-\sigma'). \tag{16}$$

For a small lattice constant a, the operator $O(\gamma)$ can be expressed approximately by

$$O(\gamma)\approx 1+a^2\frac{\delta}{\delta x_i(s)}+\frac{a^4}{2}\frac{\delta^2}{\delta x_i(s)^2}, \tag{17}$$

where i is the direction of the deformation of the string caused by the action of γ.

The kinetic energy term of the string is then rewritten as

$$\sum_\gamma{}' \tfrac{1}{2}[O(\gamma)+O^\dagger(\gamma)]=\frac{2}{a}\int ds\left(1+\frac{a^4}{2}\frac{\delta^2}{\delta x_i(s)^2}\right) \tag{18}$$

and $H_{\rm eff}$ now takes the form

$$H_{\rm eff}=\left(C_2\frac{g^2}{2a^2}-\frac{2}{a^2g^2}\right)\int ds-\frac{a^2}{g^2}\int ds\,\frac{\delta^2}{\delta x_i(s)^2}. \tag{19}$$

In the above we have accomplished the following: (1) We have traded our original field variables E, U for string variables $x(s)$, $\delta/\delta x(s)$ and replaced the original Schrödinger problem by the new Schrödinger problem Eq. (19). (2) The first term in $H_{\rm eff}$ corresponds to the electric field term and the second term to the magnetic field term in the original Hamiltonian. Hence the role of the kinetic and potential terms has been interchanged. Thus our transformation

$$A, E\to x,\ \delta/\delta x$$

is a kind of duality transformation.[4]

If we introduce an arbitrary parameter σ in place of s and denote $\vec{\varphi}(\sigma)=g^2\vec{x}[s(\sigma)]/a^2$, $H_{\rm eff}$ is further rewritten as

$$H_{\rm eff}=\int d\sigma\left[\left(\frac{\partial\vec{\varphi}}{\partial\sigma}\right)^2\right]^{1/2}\left\{\left(\frac{C_2}{2}-\frac{2}{g^4}\right)+\frac{1}{a^4}\left[\vec{\pi}-\frac{\partial\vec{\varphi}}{\partial\sigma}\left(\vec{\pi}\cdot\frac{\partial\vec{\varphi}}{\partial\sigma}\right)\left(\frac{\partial\vec{\varphi}}{\partial\sigma}\right)^{-2}\right]^2\left(\frac{\partial\vec{\varphi}}{\partial\sigma}\right)^{-2}\right\}. \tag{20}$$

$\vec{\pi}$ is canonical conjugate to $\vec{\varphi}$. This is now a nonlinear field theory in one space dimension.

It is quite interesting that we have wound up with an effective Hamiltonian defined in one spatial dimension although we started with a gauge theory in three spatial dimensions. This was made possible because we restricted outselves to gauge-invariant states of the theory which are always composed of a collection of strings. Thus it appears that the three-dimensional gauge theory may be mapped into an effective one-dimensional theory in the physical sector of its Hilbert space.

In this paper we have made some approximations in our variational wave functional, i.e., exclusion of disconnected loops and internal excitations of strings, which simplified subsequent calculations. These are approximations appropriate to QCD in its strong-coupling domain. However, even with these approximations, our effective Hamiltonian encompasses the sum of an infinite number of diagrams having the topology of a single self-nonintersecting string connecting quark and antiquark. Thus our $H_{\rm eff}$ incorporates the strong-coupling dynamics of QCD in an essential ly nonperturbative manner. When we decrease the strength of the coupling constant, diagrams with more complicated topologies will become important. The variational technique, however, would also give a reliable approximation to QCD so far as we provide enough relevant string configurations in our trial wave function. We plan to report on this problem in a future publication.

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Higher-Order Quantum Chromodynamic Corrections in e^+e^- Annihilation

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Nonleading quantum chromodynamic corrections to e^+e^- annihilation into hadrons are computed. Comparison with experiment is briefly discussed.

In quantum chromodynamics (QCD), processes which probe the structure of hadrons at short distances may be investigated with use of perturbation theory and the renormalization group.[1] The photon vacuum polarization tensor,

$$\Pi_{\mu\nu}(q) = i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(-q^2)$$

for q^2 large and spacelike, is one such short-distance probe. $\Pi(-q^2)$ and $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ can be related through dispersion relations or smearing methods.[2] In regions between new quark thresholds, where the cross section is reasonably smooth, one may hope to obtain R directly from the discontinuity of $\Pi(-q^2)$.

The leading QCD corrections to $\Pi(-q^2)$ are well known,[3] and arise from the renormalization-group improvement of the graphs of Fig. 1(a). In this paper we report a calculation of $\Pi(-q^2)$ through order g^4, arising from the graphs of Fig. 1(b). This calculation is necessary in order to determine if higher-order corrections are small, and in order that one may compare the strong coupling constant determined from measurement of R with that measured in other processes, such as deep-inelastic scattering. To address the first issue we will employ two renormalization schemes, the minimal scheme (MS) of 't Hooft and a modified scheme ($\overline{\text{MS}}$) due to Bardeen *et al.*[4] This latter scheme has been shown to lead to a more satisfactory perturbation series than MS in deep-inelastic and photon-photon scattering, and the same will be seen to be true here.

The problem of determining the strong-coupling constant can be understood in terms of the

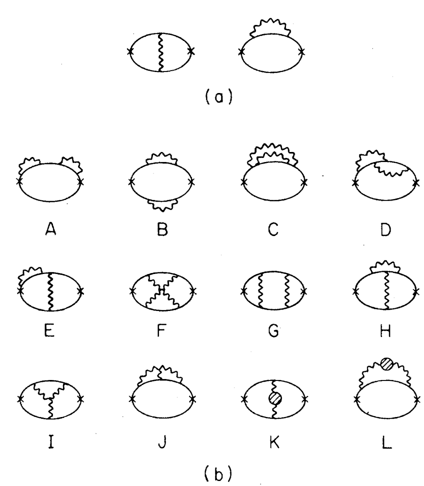

FIG. 1. (a) Graphs whose discontinuity gives R to order g^2. (b) Graphs whose discontinuity gives R to order g^4.

