New Result on the Inverse Scattering Problem in Three Dimensions

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Under the assumption that there are no bound states, a singular linear integral equation for the scattering wave function is derived whose kernel contains the scattering amplitude only. The equation is used to obtain a representation of the potential in terms of the scattering amplitude and the wave function. The final result is a three-dimensional analog of the Marchenko equation.

The problem of reconstructing the potential that causes a given scattering amplitude, if no spherical symmetry is assumed, has been attacked a number of times over many years,[1] but it has not been fully solved. I present here a new result that looks promising enough to be of interest for its potential usefulness, both in scattering theory and for possible application to nonlinear wave equations.

I shall work in three spatial dimensions and denote the unit vector in the direction of the momentum by θ. (Note that θ is not an angle.) Position vectors in R^3 will be denoted by the letters x and y, with no special vector notation, and the inner product will be written $x \cdot y$.

The scattering wave function is the unique solution of the integral equation

$$\varphi_k(\theta, x) = e^{ik\theta\cdot x} - \frac{1}{4\pi}\int d^3y \frac{e^{ik|x-y|}}{|x-y|} V(y)\varphi_k(\theta, y).$$

Let us write

$$\gamma_k(\theta, x) = \varphi_k(\theta, x) e^{-ik\theta\cdot x}. \tag{1}$$

It is well known that under very general conditions on the potential $V(y)$, $\gamma_k(\theta, x)$ is the boundary value of an analytic function of k, regular in the upper half-plane, except for poles at $k = ik_n$ if $-k_n^2$ are the bound-state energies. Furthermore, as $|k| \to \infty$ for $\mathrm{Im}k \geq 0$,

$$\gamma_k(\theta, x) = 1 + O(|k|^{-1}). \tag{2}$$

Let us assume that there are no bound states. Then it follows that $\gamma_k(\theta, x)$ satisfies the "dispersion relation"

$$\gamma_k(\theta, x) - 1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\gamma_{k'}(\theta, x) - 1}{k' - k - i\epsilon} \tag{3}$$

in the limit as $\epsilon \to 0+$.

The function $\varphi_k(\theta, x)$ is the wave function which, for $k > 0$, is usually called φ^+. The solution φ^- is related to it by

$$\varphi_k^-(\theta, x) = \varphi_{-k}(-\theta, x) = \varphi_k^*(-\theta, x), \tag{4}$$

and the two are connected by the S matrix, or the scattering amplitude.[2] This connection may be written

$$\gamma_{-k}(-\theta, x) = \gamma_k(\theta, x) + (k/2\pi i)\int d\theta' A_k^*(\theta, \theta')\varphi_k(\theta', x) e^{-ik\theta\cdot x}. \tag{5}$$

Taking the complex conjugate of (5), using (4), and the fact that (4) implies

$$A_k^*(\theta, \theta') = A_{-k}(\theta, \theta') \tag{6}$$

show that (5) holds for negative as well as positive values of k.

Let us change $k' \to -k'$ in (3) and insert (5). Then the integral becomes

$$\int_{-\infty}^{\infty} dk' \frac{1 - \gamma_{-k'}(\theta, x)}{k' + k + i\epsilon} = \int_{-\infty}^{\infty} dk' \frac{1 - \gamma_{k'}(\theta, x)}{k' + k + i\epsilon} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dk' k'}{k' + k + i\epsilon}\int d\theta' A_{k'}^*(-\theta, \theta')\varphi_{k'}(\theta', x) e^{ik'\theta\cdot x}.$$

But since $\gamma_{k'}(-\theta, x)$ is an analytic function of k' regular in the upper half-plane and it obeys (2), the first integral vanishes. Therefore (3) becomes[3]

$$\gamma_k(\theta, x) = 1 + \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{dk' k'}{k' + k + i\epsilon}\int d\theta' A_{k'}^*(-\theta, \theta')\gamma_{k'}(\theta', x)\exp[ik'(\theta + \theta')\cdot x]. \tag{7}$$

If the scattering amplitude is given, this may be regarded as a (singular) integral equation for γ. Once γ is determined the potential can be found directly from the Schrödinger equation.[4]

Equation (7) may also be used to derive a representation for the potential. The Schrödinger equation for γ reads

$$\Gamma_k(\theta, x) \equiv (\Delta + 2ik\theta\cdot\nabla)\gamma_k(\theta, x) = V(x)\gamma_k(\theta, x).$$

If (7) has a unique solution, then it follows that $\Gamma_k(\theta, x)$ must satisfy (7) with the inhomogeneity 1 replaced by $V(x)$. Therefore, applying $\Delta + 2ik\theta\cdot\nabla$ to (7) we find after a bit of algebra that[5]

$$V(x) = (i/2\pi^2)\theta\cdot\nabla\int_{-\infty}^{\infty} dk\, k\int d\theta'\, A_k^*(-\theta, \theta')\varphi_k(\theta', x)e^{ik\theta\cdot x}. \tag{8}$$

A three-dimensional analog of the Marchenko equation follows from (7) by Fourier transformation of the k dependence of the right-hand side. One readily finds that

$$\varphi_k(\theta, x) = e^{ik\theta\cdot x} + \int_{\theta\cdot x}^{\infty} d\alpha\, K(\theta, \alpha, x)e^{ik\alpha}, \tag{9}$$

where

$$K(\theta, \alpha, x) = -(i/4\pi^2)\int_{-\infty}^{\infty} dk\, k\int d\theta'\, A_k^*(-\theta, \theta')\varphi_k(\theta', x)e^{ik\alpha}. \tag{10}$$

Insertion of (9) in (10) gives

$$K(\theta, \alpha, x) = \int d\theta'\, G(\theta, \theta', \alpha + \theta'\cdot x) + \int d\theta'\int_{\theta'\cdot x}^{\infty} d\beta\, G(\theta, \theta', \alpha + \beta)K(\theta', \beta, x), \tag{11}$$

where

$$G(\theta, \theta', \alpha) = (i/4\pi^2)\int_{-\infty}^{\infty} dk\, k A_k(-\theta, \theta')e^{-ik\alpha}. \tag{12}$$

In view of (10), Eq. (8) now reads

$$V(x) = -2\theta\cdot\nabla[K(\theta, \theta\cdot x, x)]. \tag{13}$$

In order to recover $V(x)$ from the scattering amplitude, one solves (11), using (12), and inserts the result in (13). This requires a knowledge of $A_k(-\theta, \theta')$ for all k and all θ and θ'.

Generalization of these equations to the case with bound states, as well as studies of their properties, will be published elsewhere.

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[1] I. Kay and H. E. Moses, Nuovo Cimento 22, 683 (1961), and Commun. Pure and Appl. Math. 14, 435 (1961); L. D. Faddeev, Kiev University Report No. ITP-71-106E, 1971 (to be published), and Itogi Nauki Tekh., Sovren. Probl. Mat. 3, 93 (1974) [J. Sov. Math. 5, 334 (1976)]; R. G. Newton, in *Scattering Theory in Mathematical Physics*, edited by J. A. Lavita and J.-P. Marchand (D. Reidel, Dordrecht, 1974), p. 193; also lectures at the American Mathematical Society 1974 Summer Seminar on Inverse Problems, University of California at Los Angeles (unpublished).

[2] See, for example, R. G. Newton, *loc. cit.*, p. 220.

[3] This equation is the three-dimensional analog of an equation given for the Jost solution of the radial Schrödinger equation by B. R. Karlsson, Phys. Rev. D 10, 1985 (1974), and in *Applied Inverse Problems*, edited by P. C. Sabatier (Springer, New York, 1978), p. 226.

[4] Multiplying (7) by $-V(x)/4\pi$ and integrating lead directly to the well-known forward disperion relation for A.

[5] This representation of the potential may be used in the Schrödinger equation to convert it into a nonlinear equation for φ in a manner analogous to that of P. Deift and E. Trubowitz, Commun. Pure and Appl. Math. 32, 121 (1979).