Quantum-Chromodynamic β Function at Intermediate and Strong Coupling

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With use of strong-coupling methods, the energy of a long flux string is computed and this quantity is used to renormalize the theory. This gives an expansion for the renormalization-group beta function which smoothly extrapolates from the strong-coupling limit to the asymptotic-freedom value. The theory is described by three qualitatively distinct regions over this range.

The introduction of gauge-invariant lattice theories in Euclidean[1] and Hamiltonian[2] form has made it possible to study non-Abelian gauge theories directly in the phase in which quarks are confined thus achieving one of the major roles which quantum chromodynamics (QCD) is expected to play in strong-interaction physics. The other role is at short distances where the theory's asymptotic freedom[3] predicts almost free-particle behavior for quarks, with great phenomenological success.[4] A major obstacle to date has been the inability of either lattice or continuum versions to bridge the enormous qualitative gap in quark dynamics between the two. There is, however, one way in which the two kinds of quark behavior seem naturally related, and that is from the point of view of the renormalization-group beta function. Thus it is this quantity that we study in this Letter.

The lattice gauge theory is defined in terms of the partition function[1]

$$Z(\beta=1/g^2) = \int\left(\prod_l dg\right)\prod_p \exp\{\beta \operatorname{tr}[U(p)+\text{H.c.}]\}, \quad (1)$$

where l runs over all links and p over all plaquettes of a regular hypercubical lattice, dg is the normalized Haar measure over SU(3) for each link of the lattice, and $U(p)$ is the unitary $[\underline{3}]$ representation of the product of the group elements on the boundary of the plaquette p. If the transfer matrix for this partition function is constructed in timelike axial gauge and the limit that the timelike lattice spacing vanishes is taken,[5] one obtains a Hamiltonian corresponding[2] to (1)

$$H=(g^2/2a)\left\{\sum_l \vec{E}_l^{\,2} - x\sum_p \operatorname{tr}[U(p)+\text{H.c.}]\right\}, \quad x=2/g^4, \quad (2)$$

where $\vec{E}_l^{\,2}$ is the quadratic Casimir operator C_2, l runs over all links and p all plaquettes of a regular cubical lattice. Strong-coupling expansions, i.e., expansions in inverse powers of g, may be obtained from (1) and (2). For the Euclidean theory, one may use the Fourier decomposition over characters,

$$\exp\{\beta[\chi_{\underline{3}}(g)+\chi_{\underline{3}^*}(g)]\} = I(\beta)\sum_\nu \omega_\nu(\beta)\chi_\nu(g), \quad (3)$$

where ν runs over the representations of SU(3) and

$$I(\beta)=\int dg \exp[\beta(\chi_{\underline{3}}+\chi_{\underline{3}^*})]$$
$$\omega_\nu(\beta)=\int dg\ \chi_{\nu^*}(g)\exp[\beta(\chi_{\underline{3}}+\chi_{\underline{3}^*})]/I(\beta), \quad (4)$$

and group theory results on the integration of products of characters together with conventional graphical methods for high-temperature series expansions to obtain expansions for quantities of interest. For the Hamiltonian theory, one uses standard Rayleigh-Schrödinger perturbation theory in a Fock space of irreducible representations of SU(3) on each link diagonalizing $\vec{E}_l^{\,2}$. The action of products of U's on the ground state may be computed with use of SU(3) Clebsch-Gordan coefficients to decompose them. Matrix elements of gauge-invariant operators may be expressed in terms of group invariants e.g., dimensionalities, and 3-j and 6-j symbols, etc.[6]

In order to renormalize the strong-coupling expansion, we must select some dimensionful property of the theory to be held fixed. The coupling constant then becomes a function of this quantity and the cutoff, which in this case is the lattice spacing. There are no more free parameters in the theory. The quantity which we hold fixed is the coefficient of the linear term in the potential between two widely separated quarks. This is equivalent to the coefficient of the area in the expectation value of the Wilson loop, or the energy per unit length of an infinitely long string in the Hamiltonian theory. The matrix elements are to be computed in the gauge sector of the theory without the effects of quark loops which can screen the flux. There are several important advantages to this choice of normalization condition: (1) it is a natural and straightforward quantity to compute in the strong coupling expansion, (2) this condition assures that the theory always remains in the confined phase, and (3) this parameter has direct phenomenological consequences on Regge behavior and "quarkonium" spectrum. Having fixed the surface tension (or string tension, depending on circumstances), the manner in which the coupling constant varies with the cutoff is described in the usual way by the beta function:

$$\beta(g)/g = -d\ln g/d\ln a. \tag{5}$$

We report here a calculation of the surface tension in both Euclidean and Hamiltonian SU(3) lattice gauge theories to order $1/g^{20}$ in the strong-coupling expansion. The Hamiltonian result is[7]

$$\mu = (g^2/2a^2)[4/3 - (11/153)x^2 - (61/1632)x^3 - 0.012\,711\,501\,8x^4 - 0.003\,067\,187\,52x^5 - \dots],\quad x = 2/g^4. \tag{6}$$

The Euclidean result is

$$\mu = (-1/a^2)[\ln\omega + 4\omega^4 + 12\omega^5 - 10\omega^6 - 36\omega^7 + 391\omega^8/2 + 1131\omega^9/10 + 2\,550\,837\omega^{10}/512 + \dots], \tag{7}$$

where $\omega = \omega_3/3$ is the natural expansion parameter for the Euclidean theory. Differentiating (6) and (7) with respect to a gives, respectively, the results for $\beta(g)/g$ for Hamiltonian theory,

$$-\beta(g)/g = 1 - (11/51)x^2 - (183/1088)x^3 - 0.041\,378\,583\,3x^4 + 0.034\,436\,440\,6x^5 + \dots, \tag{8}$$

and, for Euclidean theory,

$$-\beta(g)/g = (2d\ln g/d\ln\omega)[\ln\omega(1 - 16\omega^4 - 60\omega^5 + 60\omega^6 + 252\omega^7 + 121\omega^8/2 + 9021\omega^9/10 - 1\,690\,677\omega^{10}/512 + \dots) + 4\omega^4 + 12\omega^5 - 10\omega^6 - 36\omega^7 + 263\omega^8/2 - 1989\omega^9/10 + 298\,037\omega^{10}/5120 + \dots]. \tag{9}$$

In Fig. 1, we present a plot of Padé approximates to the Hamiltonian beta function. The Euclidean beta function differs in details but has the same qualitative features. The asymptotic behavior for large g differs reflecting the different way in which they treat the cutoff of short distances. For $g \gtrsim 1.5$ the effects of nonleading terms in the strong-coupling expansion are small, giving a 10% or smaller correction to the beta function. For g between 1.5 and about 0.8, the higher-order strong-coupling effects become more important, driving the beta function down to the region of the weak-coupling value. Near $g = 0.9$, there is clear evidence that the Hamiltonian strong-coupling beta function is "trying to match" onto the weak-coupling function, with a sharp break from the decreasing behavior in the region 1.5 to 0.9. Below this region, where the two functions overlap, the extrapolations of the strong-coupling expansion become unreliable. At these small values of the coupling constant the perturbative expansion to the beta function is valid, and in fact the higher-order corrections to it are quite small.

Experience with previous applications of the strong-coupling expansion to the calculation of beta functions[8] leads us to expect that when further terms of the strong-coupling expansion become available the accuracy with which the extrapolated strong-coupling beta function agrees with the weak-coupling result will improve. Nevertheless, the current degree of agreement is remarkable.

These results have several implications for physics. For the first time, there is clear evidence that strong-coupling methods may be capable of reproducing the asymptotic freedom which is necessary to describe the short-distance properties of hadrons. By combining the weak- and strong-coupling results which have an overlap region where they are both valid, one obtains

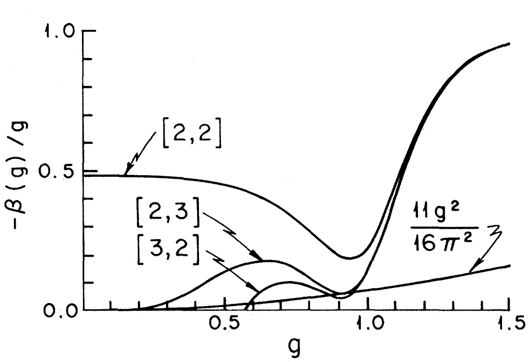

FIG. 1. Padé approximates to the Hamiltonian beta function.

for the first time a quantitative description of the theory over the entire range of coupling constants. In fact, we may use the weak-coupling result for the beta function to extrapolate the string tension to zero lattice spacing, which gives, after renormalization, a value for $g(a)$ for all a. If we choose a physical value for the string tension, we obtain a zero-parameter approximation to $\alpha = g^2/4\pi$ since $g(a)$ is an approximation to the renormalized coupling constant at a length scale of a.

There is a clear separation of the physics into three regions. As previously mentioned, for $g \lesssim 0.8$ weak coupling prevails; for $0.8 \lesssim g \lesssim 1.5$, there is a transition region between weak and strong coupling; and for $g \gtrsim 1.5$, one is in a strong-coupling regime. From the point of the strong-coupling expansion, the onset of the transition is caused by the thawing of fluctuations in the string, allowing the flux to spread out lowering the energy. From the weak-coupling end, the onset of the transition cannot be perturbative in origin since the higher-order corrections to the beta function[9] are far too small to account for it. However, estimates of the effects of instantons on the weak-coupling beta function made by Callan, Dashen, and Gross[10] give a correction which has all the desired qualitative features. Thus it appears that the origin of the transition from the weak-coupling point of view is the effect of vacuum tunneling, i.e., gauge fluctuations which begin to trap the flux. Just the opposite of the effect which causes the transition from the strong-coupling end. So we see that the main characteristic of the transition region is the smooth transition from flux confined to strings to unconfined flux over a narrow range of coupling constant.

A lattice theory is of course only an approximation at any finite lattice spacing and in order to recover a complete physical description one must be able to take the limit that the lattice spacing vanishes. During most of this process the coupling will be quite small and essentially one may use weak-coupling methods. However, one begins in the strong-coupling regime and must be able to safely make the transition to the weak-coupling regime. The importance of the present result is that it demonstrates that this is possible with only a few orders in the strong-coupling expansion. Several further areas of exploration are suggested. A combination of strong-coupling Padé methods with the renormalization group at weak coupling as suggested by Wilson[11] may prove to be very powerful. It is also of interest to study the strong-coupling expansion for a modified Hamiltonian with the leading irrelevant operators removed by adding six link terms. And, of course, efforts are under way to extend the strong-coupling series to higher orders. We feel that it should be possible to eventually compute corrections to order $1/g^{36}$ using present methods. (The calculation of the g^{-24} coefficient is nearing completion.)

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[7] The exact coefficients are

$$-\frac{737\,327\,120\,374\,220\,449}{58\,004\,722\,308\,819\,686\,400}$$

$$-\frac{98\,631\,094\,843\,173\,218\,126\,309}{32\,156\,851\,302\,637\,820\,478\,720\,000}.$$

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