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Instability of Phase Coexistence and Translation Invariance in Two Dimensions

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All the equilibrium states of the two-dimensional Ising system are convex combinations of the pure phases and, as such, translation invariant. This has been rigorously proven and is related to the instability of phase boundary lines with respect to thermal fluctuations. While in higher dimensions stable phase coexistence is known to occur, the instability is expected to be typical of two-dimensional systems.

In order to improve the understanding of the cooperative phenomena present in bulk systems, a great deal of attention has been given in statistical mechanics and solid-state physics to models of locally interacting spins on a lattice. A generic feature of such systems is the discontinuity, at certain values of the relevant parameters (which represent the temperature, magnetic field, etc.), of the states of thermodynamic equilibrium. At these values, the system may exist in various phases. A pertinent question is then the possibility of a locally stable phase coexistence. It would be described by an equilibrium state in which in different regions the spin configurations show behavior typical of different phases. In some cases, as the one discussed below, the coexistence may be described by the presence of a sharply defined interface. Further, if both the interactions and the pure phases are translation invariant, the phase coexistence relates to the possibility of the breaking of translation symmetry in the infinite volume limit.

It is generally expected that the stability of phase coexistence is dimension dependent, as many other phase transition phenomena are known to be. One method to induce phase coexistence, introduced by Dobrushin in 1972,[1] is by mixed boundary conditions, i.e., of one type on the upper halves of the surfaces of a sequence of cubes and of the other type on the lower halves. In three or more dimensions such boundary conditions for the Ising system at low temperatures were proven to produce states in which there is a basically horizontal interface.[1] The proof that for these states translation symmetry in the vertical direction is broken was much simplified by von Beijeren.[2] The construction was recently extended to the Widom-Rowlinson lattice model.[3]

In contrast, the fluctuations of the analogous contour in the two-dimensional Ising system with the above boundary conditions are unbounded when the size of the box is increased to infinity. For low temperatures, it has been shown,[4] that in this limit any fixed finite region would typically be deep inside a region which is surrounded by either +1 or -1 spins. In such a case, the limiting state is $(\mu_+ + \mu_-)/2$, i.e., an ensemble average of the two pure phases. Subsequently, with the aid of the explicit calculations of Abraham and Reed,[5,6] this structure of the limiting state was proven for all the temperatures below T_c and various other boundary conditions.[7] These results raise the possibility that the only Gibbs states of the two-dimensional system are the convex combinations of pure phases. The best evidence in this direction has been a recent proof, by Russo,[8] of such assertion for all the Gibbs states which have one of the main symmetries of the lattice.

The main result described here finally settles the question for the two-dimensional Ising system. I give an outline of a rigorous proof that at any temperature below T_c, if the system is locally in a thermodynamic equilibrium, its state is an ensemble average of the two pure phases. By implication, when contact is attempted between two phases, the fluctuations would spread one of them over the other. Another conclusion is that any of the system's "Gibbs states" has the full symmetry of the lattice. Extensions to other systems are briefly mentioned.

Statement of the main result.—In the Ferromagnetic Ising System (FIS), the spin variables σ_i, which are associated with the lattice sites, $i \in Z^d$, take the values ± 1 with equal *a priori* probabilities. The Hamiltonian is

$$H = \sum_{|i-j|=1} \sigma_i \sigma_j .$$

In analogy with finite systems, states in the thermodynamic limit (i.e., infinite volume) may be described by probability measures on the space of spin configurations, $\Omega = \{-1, 1\}^{Z^d}$. States which are limits of equilibrium ensembles of finite systems, with some boundary conditions, are characterized by the Dobrushin-Lanford-Ruelle (DLR) equations[9] (which are the variational equations for local minima of the free energy) and are called Gibbs states.

If μ and ν are two probability measures on Ω we say that μ dominates ν in the Fortuin-Kasteleyn, and Ginibre (FKG) sense ($\mu \underset{\mathrm{FKG}}{\geq} \nu$) if, denoting expectation values by the same symbols as the measures, $\mu(f) \geq \nu(f)$ whenever f is a monotone increasing function of the spins.[10] A very useful property of the FIS, to which I shall refer as the FKG property, is that whenever a configuration of spins on a boundary of a set pointwise dominates another spin configuration, then the Gibbs state induced by the first dominates, in the FKG sense, the state induced by the other.[11]

The FKG property of FIS implies that for any temperature the (unique) maximal and minimal Gibbs states, in the FKG sense, are obtained by taking the limits of grand canonical ensembles with $+1$ and -1 boundary conditions on any monotone sequence of sets which increase to z^d. We denote these states by μ_+, μ_-. The FKG property also implies that the following are equivalent conditions: (1) $\mu_+ \neq \mu_-$; (2) $\mu_{(\overset{+}{-})}(\sigma_0) \gtrless 0$; and, for $d=2$,[8,12] (3) typical (i.e., "almost all") spin configurations for μ_+ (μ_-) have an infinite cluster of spin $+1$ (-1). [The implication (1)$\Rightarrow$(3) is true for any d.] By a cluster is always meant a subset of the lattice which is connected in the nearest neighbor sense, and a *cluster a set which is connected in the weaker sense which permits diagonally nearest neighbors.

At a high temperature $\mu_+ = \mu_-$; however, if $d \geq 2$ there is a transition temperature, T_c, below which the conditions (1)–(3) are satisfied. At these temperatures there are two distinct translation-invariant extremal states, which are the pure phases.

My main result is that in two dimensions any Gibbs state of the FIS is of the form $\lambda\mu_+ + (1-\lambda)\mu_-$, for some $\lambda \in [0,1]$. Some of the implications were discussed above. Full details of the argument and its extension to other two-dimensional systems will be given elsewhere.[12]

Outline of the argument.—The first step in the analysis is to identify a convenient feature which may be referred to as the interface. In two dimensions, for $T < T_c$, this is simplified by the absence, in configurations which are typical for μ_+ or μ_-, of infinite *clusters of the opposite signs.[8,10] Thus, such configurations do not have infinite contours, which are nonintersecting lines which separate sites of different spins. The ambiguity in the association of closed contours with spin configurations is resolved by a simple convention which we adopt from Ref. 4. The infinite contours meet, therefore, two requirements of an interface: (i) They are absent in pure phases; (ii) for the regions enclosed by them they provide boundary conditions which are known to lead to pure phases. The third, and for us the most important, requirement is this: (iii) the absence of an interface should indicate that the state is an ensemble average of the pure phases. This condition is also satisfied, as may be proved with use of the FKG property. It leads to the following criterion:

Criterion A: Let μ be a Gibbs state of the FIS. Then $\mu = \lambda(1-\lambda)\mu_+ + \lambda\mu_-$ for some $\lambda \in [0,1]$, if, and only if, μ-almost every spin configuration has no infinite contour.

In view of the above result, I adopt the infinite contours as defining the interface lines. This permits the application of geometrical ideas about the fluctuations of such lines to the analysis of general Gibbs states. The new technique is best exemplified in the proof of the next step of the argument, in which I rule out what seems to be the simplest mode of phase coexistence.

Claim B: There is no Gibbs state with respect to which there is a nonvanishing probability for the existence of a single infinite contour which, further, has finite and nonempty intersections with the lines $\{k\} \times Z$, $\forall k \in Z$.

Before outlining the argument, let me point out that the decompsition of the state implied by Criterion A corresponds to partitioning the space of spin configurations by the sign of their infinite cluster. This is an example of a partition which is "measurable at infinity" since its outcome is not affected by any local change of spins. It is very useful to know that any Gibbs state can be decomposed (uniquely) as a convex combination of extremal Gibbs states, for which the asymptotic behavior of the typical configurations is the same. More concretely, for extermal Gibbs states the probability of any set which is measurable at infinity is either 0 or 1.

To prove claim B, it is enough, therefore, to disprove the existence of an extremal Gibbs state for which almost surely (i.e., with probability one) there is a unique interface $\gamma(\sigma)$, with the above properties. Let μ be such a state. Since the lowest intersection of γ with the "y axis" is well defined, it has some probability distribution. This is contradicted by proving that μ has to be invariant with respect to translations in the "y direction." Following is an outline of the proof.

Let $\hat{\mu}$ be the state obtained by shifting μ one step in that direction [i.e., $\hat{\mu}(d\sigma_i) = \mu(d\sigma_{i-(0,1)})$]. In order to compare the two, we sample pairs of spin configurations $(\sigma, \hat{\sigma}) \in \Omega \times \Omega$ distributed independently with the probability $\mu \times \hat{\mu}$. To $\mu \times \hat{\mu}$-almost every such pair there correspond two infinite contours $\gamma = \gamma(\sigma)$ and $\hat{\gamma} = \gamma(\hat{\sigma})$. The levels of intersections of these contours with vertical lines have to have some (thermal) fluctuations. Thus, even though the distribution of $\hat{\gamma}$ is shifted by one step upward with respect to γ, it is no surprise that one may prove (using the above 0–1 law for events measurable at infinity) that

(i) $\mu \times \hat{\mu}$-almost surely, γ and $\hat{\gamma}$ intersect infinitely often.

By the extremality of μ, the infinite clusters below γ and $\hat{\gamma}$ are almost surely of the same signs and, since the infinite contour is unique, below γ and $\hat{\gamma}$ there are no infinite clusters of the opposite sign. Therefore, by the assertion in (i) it is not possible to find infinite clusters on which $\sigma > \hat{\sigma}$. This implies that any finite volume may be completely surrounded by a connected set on which $\sigma \leq \hat{\sigma}$. By the FKG property and the Markov property of the DLR condition this leads to $\mu \underset{\text{FKG}}{\leq} \hat{\mu}$. For a similar reason, $\mu \underset{\text{FKG}}{\geq} \hat{\mu}$, which implies that (ii) $\mu = \hat{\mu}$. Hence, as claimed above, μ is translation invariant, which leads to Claim B.

Next is a reduction of the general case which is proved by a refinement of the argument of Ref. 8.

Claim C: The number of infinite contours in typical configurations is at most one, for any Gibbs state below T_c. If they occur, the contours have the properties assumed in Claim B.

The only possibility left open by the above two claims is the one to which the Criterion A applies. This proves the main assertion.

Further extensions.—The main result discussed here may be a generic feature of two-dimensional short-range systems. Most of the above arguments extend to some other models which have the FKG property, like the Widom-Rowlinson lattice model.[3,13] The remaining obstacle, which is a much weaker version of Claim B, may be overcome by a Peierls-type argument. This leads to a proof of an analogous result subject, however, to the restriction to low temperatures.

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