an (1). However, this term is a redundant variable. By applying an appropriate rotation in the (φ_1, φ_2) plane, the Hamiltonian (13) can be transformed into a model of the form given by Eq. (1). By applying uniaxial and shear stresses along various direction in the x-y plane, the fields g_1 and g_2 are realized and the (g_1, g_2, T) phase diagram can be mapped.

The analysis presented in this paper can be easily extended to the case of the $n=3$ cubic $RbCaF_3$, and the phase diagram (2) is expected to be realized in this crystal. It is also expected that similar phase diagrams should also occur in systems with no stable fixed point, such as UO_2, MnO, Cr, and Eu. This can be achieved by applying a symmetry-breaking field which favors an ordering different from the one favored by the fourth-order anisotropic terms.[15]

In summary, we have demonstrated that the crossover from first order to continuous phase transition induced by symmetry-breaking fields can lead to quite complicated and interesting phase diagrams. We suggest that these phase diagrams be tested experimentally in real physcial systems.

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Gauge Wheel of Superfluid ^{3}He

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A change in the phase of the order parameter of ^{3}He-A can be undone by a subsequent rotation. We investigate the dynamical consequence of this broken relative gauge and rotational symmetry. In particular a wheel rotated in the liquid acts as a "gauge transformer" driving a superflow. Such experiments provide a very direct probe of this unusual feature of the order parameter, at the same time measuring the orbital quantum number of the pairs.

The defining property of a superfluid is the broken gauge symmetry; that is, the phase of a wave function becomes a significant and macroscopic variable. The Anderson-Brinkman-Morel (ABM) state, generally accepted to describe the ^{3}He-A phase,[1] breaks this gauge symmetry in a rather special way: A phase transformation followed by a rotation through the same angle around the preferred direction $\hat{l}$ leaves the order parameter unchanged. This well-known symmetry has rather intriguing dynamical consequences, which we wish to examine in this paper. We shall provide a rigorous, yet very simple, proof that whenever the orbital part of the pair wave function is a pure angular momentum state, characterized by the two quantum numbers L and M_L ($L = M_L = 1$ for the ABM state), the Josephson equation is modified to include a term proportional to the vorticity $\vec{\Omega} = \frac{1}{2}\nabla\times\vec{v}_n$:

$$d\varphi/dt + M_L \hat{l}\cdot\vec{\Omega} + (2m/\hbar)\mu = \text{dissip. terms}. \quad (1)$$

We shall argue that this term represents the novel aspects of the A-phase dynamics—i.e., those which are not simply the superposition of He II and nematic liquid crystal dynamics. Furthermore, since the existence of this term and the microscopic information it contains are independent of any specific models (such as weak coupling), experiments verifying Eq. (1) constitute a very direct way of identifying the orbital state, and incidentally are probing quantum mechanics on a macroscopic scale. We shall, at the end of the paper, propose such an experiment which we believe is feasible with today's technology.

The structure of the hydrodynamic theory depends only on the quantities conserved and symmetries broken. Therefore, superfluid ^{3}He, breaking the same continuous symmetries that are broken one at a time in antiferromagnets, He II, and nematic liquid crystals, displays a rich behavior with all the characteristics of these three systems. However, ^{3}He is not only complex, but also unique: Adding a twist to the principle of spontaneously broken symmetry, it breaks the various symmetries in a special way, resulting in novel dynamic properties, which we believe are especially worthy of attention.

Each of the two bulk stable states of superfluid ^{3}He breaks *a linear combination of two continuous symmetries*, while a second, independent combination remains an invariant operation. The B phase breaks the well-known spin-orbit symmetry, and the A phase breaks the difference between the gauge transformation and the rotation around $\hat{l}$. We may refer to it as the relative gauge-orbit symmetry or, for brevity, GOS. Since the second combination (given by the sum) remains an invariant operation, ^{3}He-A cannot distinguish between the two underlying symmetry transformations. This has intriguing consequences. A simple *Gedankenexperiment* will illustrate this effect. Consider a disk immersed in helium with $\hat{l}$ uniform everywhere. (We shall unrealistically neglect what happens close to and beyond the edge of the disk to simplify the arguments.) Turning the disk at a constant angular velocity will drag the viscous normal fluid along. A nonvanishing vorticity is equivalent to a local rotation, and so its component along $\hat{l}$ will "wind up" the phase and maintain a countercurrent of superflow and normal flow. Depending on the sense of rotation, the region close to the disk will thus be either cooled or warmed up. One can alternatively insert a superleak in the flow path; then instead of a temperature drop a fountain pressure will be built up. Similarly, an oscillating disk will, depending on the geometry, radiate second or fourth sound. These effects clearly set ^{3}He-A apart from other known superfluids.

We proceed to rederive the A-phase orbital hydrodynamics,[2-4] concentrating on the Josephson equation (1). Any system in which the gauge and the three orbital symmetries are independently broken is characterized by the four additional equations of motion[5] of the phase φ^0 and the infinitesimal rotation vector $d\vec{\theta}^0$:

$$d\varphi^0/dt = -2m\mu/\hbar, \quad d\vec{\theta}/dt = \vec{\Omega}. \quad (2)$$

These two equations[6] in conjunction with the conservation laws of the energy, density, and the three components of momentum constitute the complete hydrodynamic theory. In a system which breaks GOS and two orbital symmetries, however, one linear combination of the above four equations is irrelevant, because the corresponding symmetry transformation is an invariant operation. More specifically, when a pair wave function is a pure angular momentum state, the symmetry operation given by a simultaneous gauge transformation of an angle α and a rotation through α/M_L around $\hat{l}$ is an invariant one. When we discard this linear combination, the three remaining ones $d\vec{\theta} = d\vec{\theta}^0 - \hat{l}\cdot d\varphi^0/M_L$ constitute the relevant broken symmetry variables, and we need only consider the equation of motion[6]

$$d\vec{\theta}/dt = \vec{\Omega} + \hat{l}\,2m\mu/\hbar M_L. \quad (3)$$

Note that it is $d\vec{\theta} = \hat{l}(\hat{\Delta}_2\cdot d\hat{\Delta}_1) + \hat{l}\times d\hat{l}$ and not $d\vec{\theta}^0$ which describes the rotation of the order-parameter triad, conventionally given by $\hat{\Delta}_1$ and $\hat{\Delta}_2$. Rotation vectors are only infinitesimally well defined, and therefore in manipulating $d\vec{\theta}$ in the presence of global texture, one has to pay attenution to the commutation relation

$$(\nabla\delta - \delta\nabla)\vec{\theta} = \nabla\vec{\theta}\times\delta\vec{\theta}, \tag{4}$$

where ∇ and δ stand for any first-order differential operator such as $\partial/\partial t$. The connection to the usual and more redundant notation is easily established: One can divide the rotation vector into two parts, the rotation around $\hat{l}$ and the rotation of $\hat{l}$, being, respectively,

$$-d\varphi/M_L = \hat{l}\cdot d\vec{\theta}, \quad d\hat{l} = d\vec{\theta}\times\hat{l}, \tag{5}$$

with the superfluid velocity expressed as $\vec{v}_s = -M_L(\hbar/2m)l_i\nabla\theta_i$. In these variables, Eq. (3) splits naturally into the modified Josephson equation (1) and the equation of motion for $\hat{l}$ [and Eq. (4) becomes the Mermin-Ho relation[7]]. This completes the simple proof that Eq. (1) is a direct consequence of broken GOS. The full set of nonlinear hydrodynamic equations can now be derived with the help of the standard procedure[8] with no new insights. One point, however, is worth mentioning. Because of the Onsager relation for canonically conjugate variables, the coupling of $\dot{\varphi}$ to $\vec{\Omega}$ in Eq. (1) leads to a term in the stress tensor[3] that is equivalent to an angular momentum current $-M_L(\hbar/2m)l_ij_j{}^s$. This is very plausible when we note that in $\partial\epsilon/\partial\nabla\theta_{\parallel}{}^0 = -M_L\partial\epsilon/\partial\nabla\varphi^0$ (where ϵ is the energy density and $\parallel$ refers to $\hat{l}$), the left-hand side can be interpreted as a torque, whereas the right-hand side (multiplied by $2m/\hbar M_L$) is the usual expression for the supercurrent. In this instance at least the superfluid fraction behaves as if it carries an angular momentum $M_L\hbar\hat{l}$ per pair of particles. Also, it is tempting to view the $M_L\hat{l}\cdot\vec{\Omega}$ term in Eq. (1) as an additional chemical potential arising from the coupling of the pair angular momentum to an external rotational velocity $\vec{\Omega}$, an energy $\frac{1}{2}M_L\hbar\hat{l}\cdot\vec{\Omega}$ per particle.

The modification of the Josephson equation with $M_L = 1$ was first proposed by one of us[3] on the ground that it is permitted by symmetry and, in fact, is required if one assumes that the order-parameter triad follows the rotation of v_n in all three directions. Subsequently, Hu and Saslow, and Ho,[4] showed that the term $\hat{l}\cdot\vec{\Omega}$ can also be derived by requiring local conservation of angular momentum. Its existence, however, remains controversial. In two recent publications,[9] the microscopically derived Josephson equation either does not contain it at all or contains it in combination with the Yosida function. In view of the above considerations, establishing the rigorous connection between this term and broken GOS, further scrutiny of the microscopic results seems necessary.

Next we turn our attention to the verification of Eq. (1) via the phase-winding effect. From the definition of v_s and use of Eq. (4) it can be easily deduced that an infinitesimal rotational transformation $d\vec{\theta}$ of an arbitrary texture generates a velocity change:

$$d\vec{v}_s = -M_L l_i\nabla d\theta_i. \tag{6}$$

Hence, to demonstrate the effect, we need a rotation that is spatially varying[10] and is along $\hat{l}$. These two points provide the motivation for the following geometry, which utilizes the phase winding to excite a fourth-sound resonance between two vessels.[11]

A thin cavity (height $H\parallel\hat{z}$) containing ^{3}He has a pronounced elliptical shape (axes $a\parallel\hat{x}$, $b\parallel\hat{y}$ with $b\gg a\gg H$) and a little hollow tail (length $L\parallel\hat{x}$, width $w\parallel\hat{y}$, and height $h\parallel\hat{z}$, $L\gg w\gg h$) along $\hat{x}$. The internal volume of the tail V_T is much smaller than that of the elliptical cavity V_E. The other end of the tail is connected to a box (helium volume $V_B\simeq V_E$), filled with sintered copper and in contact with the refrigerator. The whole system (I) is joined to another mirror-symmetric one (II) by n very thin tubes (length l and radius $r\ll H$ or h), such as given in a glass capillary array, between the two boxes. The dimensions of H, h, and especially r are such that the normal fluid is very well clamped and $\hat{l}$ is predominantly $\parallel\hat{z}$ in V_E and V_T and perpendicular to the axis in the tubes. The two cavities are driven to oscillate in the plane $\perp\hat{z}$ with the boxes and the tubes kept fixed. Since r is by far the smallest dimension of the apparatus, there are two very different equilibrating times. On the scale of the fourth-sound oscillation between I and II, we can assume equilibrium within I (or II), and Eq. (1) (under the condition of uniform $\hat{l}$ texture) becomes $\mu_{\rm I} = \mu(r) + (M_L\hbar/2m)\hat{l}\cdot\vec{\Omega}$, with $\mu_{\rm I}$ = const throughout I. The change in chemical potential from its $\Omega = 0$ value $\mu_{\rm I}{}^0$ is achieved by moving some superfluid mass within I; hence $\int\mu_{\rm I}{}^0 d^3r = \int\mu(r)\,d^3r$ (integrated over $V_{\rm I} = V_E + V_T + V_B$), and we have

$$\mu_{\rm I} = \mu_{\rm I}{}^0 + (M_L\hbar/2m)\Omega_{\rm I}G, \tag{7}$$

where $\Omega_{\rm I}$ is the angular velocity of the elliptical cavity I and $G = \int\hat{l}\cdot\vec{\Omega}\,d^3r/\Omega_{\rm I}V_{\rm I}\simeq\frac{1}{2}$ is an apparatus constant depending only on its geometry and elastic properties. If we take $\Omega_{\rm I} = \Omega_{\rm II} = \Omega_0 e^{i\omega t}$ if $\hat{l}$ in I and II are parallel and $\Omega_{\rm I} = -\Omega_{\rm II} = \Omega_0 e^{i\omega t}$ if they are antiparallel, the difference in the chemical potential across the tube will drive a superfluid current, $\dot{v}_s = (\mu_{\rm I} - \mu_{\rm II})/l$, which in conjunction with the continuity equation, $\dot{\rho}_{\rm II} - \dot{\rho}_{\rm I} = 2n(\pi r^2) \times \rho_\perp{}^s v_s / V_{\rm I}$ (ρ is the mass density), gives rise to a fourth-sound resonance with the velocity $c_4 = (\rho_\perp{}^s\, \partial\mu/\partial\rho)^{1/2}$ and the effective wave vector $q^2 = 2n\pi r^2 / l V_{\rm I}$:

$$\delta\mu = \frac{2G(\hbar/2m)M_L\Omega_0 e^{i\omega t}}{1 - \omega^2/c_4{}^2 q^2 + i/Q}. \qquad (8)$$

For $r \simeq 1\,\mu$m, the Q value $Q = 1/\omega\tau_4$ is essentially determined by the viscous compression of the superfluid, for which Wölfle[12] estimated $\tau_4 \simeq 10^{-7}$ s. The loss due to the normal fluid being squeezed through the tubes according to the Hagen-Poiseuille law is about ten times smaller. Neglecting also the tiny thermal contribution, the on-resonance pressure is given by $\delta p = \rho\delta\mu = M_L \times 10^{-4}\theta_0 \times e^{i\omega t}$ bars. Further scrutiny shows that an angular displacement of $\theta_0 \simeq 10^{-3}$ in combination with $\omega \simeq 200$ s^{-1} generates virtually no interfering side effects. This leads to a measurable pressure change, whose value will also yield information on M_L.

There are three types of conceivable nuisance effects. The first one is given by the dissipation connected to the driving such as the normal slip in the cavity or the forced shearing in the tail. They do not lower the Q value but do generate heat, though only of the order of 10^{-14} W. The second type is given by textural disruptions, when v_s (with respect to v_n) exceeds some critical value in the ellipses, tails, or tubes. An example is the Fréedericksz transition.[13] With the ellipses having the largest dimension and the highest angular velocity, it is most likely to happen there first. However, a cavity with a pronounced elliptical shape and a rotational velocity Ω can be shown[14] to have the relative superfluid velocity $v_x{}^s = 2\Omega y a^2/b^2$, $v_y{}^s = -2\Omega x$, and therefore a magnetic field ($\gtrsim 100$ G) along $\hat{y}$ should be quite effective in suppressing a textural transition. The on-resonance velocity in the glass capillary array can be quite large. With the standard dimension $l = 3$ mm, however, we have $v_s \lesssim 0.2$ mm/s, which is far smaller than the calculated value (3 cm/s) of the textural critical velocity.[15] The third one is related to the nonuniform $\hat{l}$ texture at the edges of V_T that run along $\hat{x}$ and may also drive the resonance. This effect, however, vanishes with the ratio of the nonuniform area to the total cross section, designed to be small.

A variation of the above geometry is given by replacing the tail with a cylindrical cavity (radius R, length L, $L \gg R$), which connects the center of the ellipse with the cooling box along $\hat{z}$. For $R \simeq 100$ μm we can expect the cylinder to have the Anderson-Brinkman[1] (or Mermin-Ho[7]) texture. Bending the tail is thus equivalent to twisting the tube and all the above considerations remain valid. Although this setup is easier to drive and results in less damping, there is also a drawback: The magnetic field employed to prevent textural disruption will distort the cylindrically symmetric texture and may give rise to an additional driving mechanism.

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