Internal-Wave Solitons of Fluids with Finite Depth

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A nonlinear internal wave equation that describes stratified fluids with finite depth has been studied. N-soliton solutions were found through Hirota's method. Although the equation tends to either the Korteweg–de Vries equation or the Benjamin-Ono equation in the shallow- or deep-fluid limit, respectively, the N-soliton solutions obtained tend to the Korteweg–de Vries solitons in the shallow-fluid limit but do not tend to the Benjamin-Ono solitons in the deep-fluid limit. Therefore, there is no smooth transition from one kind of soliton to another with varying depth of the fluid.

It is well known that nonlinear internal waves propagating along the interface of two fluids with different densities can be modeled by two nonlinear wave equations.[1-5] In the deep-fluid limit, they are described by the Benjamin-Ono equation,

$$q_t + 2qq_x + Hq_{xx} = 0, \tag{1}$$

where H is the Hilbert transform operator defined by

$$Hq(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{q(z)}{z-x}\,dz; \tag{2}$$

while in the shallow-fluid limit, they are described by the Korteweg–de Vries (K-dV) equation,

$$U_t + 2UU_x + U_{xxx} = 0. \tag{3}$$

The latter equation (3) is the first equation known to possess multisoliton solutions. The inverse-scattering method[6] has been developed to obtain its complete solution. The former equation (1) does not yet have a complete solution. But a pole-expansion method can be applied to obtain general N-soliton solutions.[5] Unlike the Korteweg–de Vries solitons that are of squared hyperbolic secant type (nonalgebraic),

$$U = k^2 \operatorname{sech}^2 k(x - k^2 t), \tag{4}$$

the Benjamin-Ono solitons are algebraic (or rational),

$$q = 2v/[v^2(x-vt)^2 + 1]. \tag{5}$$

A general property of algebraic solitons in the absence of phase shift after collisions of two such solitons. Therefore, algebraic solitons are truly independent nonlinear normal modes of a system which would show perfect recurrence instead of an approximate one.

Recently, Joseph[4] studied the problem of nonlinear waves in a fluid of finite depth. In case of a thin thermocline located at the depth $z = -d$ in a fluid of total depth D, the equation is[4]

$$\varphi_t + 2\varphi\varphi_x + \partial_x \int_{-\infty}^{\infty} \varphi(x', t) G(x' - x)\,dx' = 0, \tag{6}$$

with

$$G(x) = (c_0/2\pi)\int_{-\infty}^{\infty} dk[1 - \tfrac{1}{2}kd(\coth kD)]e^{ikx}.$$

This equation has a linear dispersion relation given by

$$\omega = kc_0[1 - \tfrac{1}{2}kd(\coth kD)]. \tag{7}$$

In the deep-fluid limit, $D\to\infty$, we get

$$\omega \to kc_0(1 - \tfrac{1}{2}d|k|),$$

the Benjamin-Ono dispersion,[1] and Eq. (6) would reduce to (1) (if we disregard the irrelevant translation term and unimportant coefficients). However, in the shallow-fluid limit, $D\to 0$, we get

$$\omega \to \tfrac{1}{2}kc_0(1 - \tfrac{1}{3}k^2 dD),$$

the Korteweg–de Vries dispersion, and Eq. (6) would reduce to (3).

Joseph[4] claimed that he had found a single-soliton solution of Eq. (6) which could be reduced to either the Benjamin-Ono or the Korteweg–de Vries solitons in the respective limit. It is therefore interesting to investigate his claim and see whether N-soliton solutions also exist for Eq. (6). In this paper, we shall demonstrate the existence of multisoliton solutions of Eq. (6). However, we shall also show that these soliton solutions, although they tend to the K-dV solitons in the shallow-fluid limit as expected, do not tend to the Benjamin-Ono solitons in the deep-fluid limit. There is no smooth transition as Joseph claimed. On the other hand, the generalized solution that would reduce to the Benjamin-Ono solitons in Eq. (6) has not been found. In fact, it may not exist at all. Nevertheless, as we shall demonstrate, Eq. (6) possesses a rich set of regular solutions including nonalgebraic soliton solutions, multiperiodic solutions, and also algebraic soliton solutions. To demonstrate these solutions, we rewrite Eq. (6) as a differential-difference equation

$$\varphi_t + 2\varphi\varphi_x - (C/S)\varphi_{xx} = 0, \tag{8}$$

where C and S are difference operators along the imaginary axis such that

$$C\varphi(x) = \varphi(x+iD) + \varphi(x-iD) \equiv \varphi_+ + \varphi_-,$$
$$iS\varphi(x) = \varphi(x+iD) - \varphi(x-iD) = \varphi_+ - \varphi_-. \tag{9}$$

In so doing, we have again neglected some irrelevant terms and coefficients. In the following, we shall concentrate on the solutions of Eq. (8).

In obtaining solutions of Eq. (8), we find Hirota's method[7] the most straightforward. Substituting

$$\varphi = S\partial_x \ln f = -i(f_{+,x}/f_+ - f_{-,x}/f_-), \tag{10}$$

into (8), we obtain a bilinear equation for f:

$$i(f_{+,t}f_- - f_{-,t}f_+) + f_+f_{-,xx} + f_-f_{+,xx} - 2f_{+,x}f_{-,x} = 0. \tag{11}$$

A single-soliton solution is then given by

$$f(x,t) = 1 + e^{2(kx+\omega t+\eta)}, \tag{12}$$

where $\omega = 2k^2\cot 2kD$, and η is a real constant, or

$$\varphi_S(x,t) = -ik\{\tanh[k(x+iD)+\omega t+\eta] - \tanh[k(x-iD)+\omega t+\eta]\} = \frac{2k\sin 2kD}{\cosh 2(kx+\omega t+\eta) + \cos 2kD}. \tag{13}$$

It is interesting to note that this soliton can move either to the left or to the right depending on the magnitude of parameter k. We note also that this solution approaches the K-dV soliton in the limit $D \to 0$:

$$\varphi_S \xrightarrow{\text{K-dV}} 2k^2D\,\mathrm{sech}^2 k\left[x + \left(\frac{1}{D} + \frac{4k^2D}{3}\right)t + \frac{\eta}{k}\right].$$

However, if we were to try taking the $D \to \infty$ limit, we should not recover the Benjamin-Ono soliton. In fact, there is no proper limit in this case. On the other hand, a periodic solution[8] may be obtained by the replacement $k \to i\bar{k}$, $\eta \to i\bar{\eta}$ in solution (13):

$$\varphi_p(x,t) = -2\bar{k}\sinh 2\bar{k}D/[\cos 2(\bar{k}x + \bar{\omega}t + \bar{\eta}) + \cosh 2\bar{k}D]. \tag{14}$$

In the $D \to 0$ limit, it approaches

$$\varphi_p \xrightarrow{\text{K-dV}} 2\bar{k}^2D\sec^2(\bar{k}x + \bar{\omega}t + \bar{\eta}), \tag{15}$$

a singular periodic solution [note that (14) is regular] of the K-dV equation.

A different set of regular solutions can be obtained by letting $\eta \to \eta + \frac{1}{2}i\pi$ and we have

$$\bar{\varphi}_S = -ik\{\coth[k(x+iD)+\omega t+\eta] - \coth[k(x-iD)+\omega t+\eta]\} = \frac{2k\sin 2kD}{-\cosh 2(kx+\omega t+\eta) + \cos 2kD}. \tag{16}$$

Its K-dV limit is

$$\bar{\varphi}_S \xrightarrow{\text{K-dV}} 2k^2D\,\mathrm{csch}^2(kx+\omega t+\eta), \tag{17}$$

again a singular solution. The periodic solution is obtained similarly,

$$\bar{\varphi}_p = 2\bar{k}\sinh 2\bar{k}D/[\cos 2(\bar{k}x + \bar{\omega}t + \bar{\eta}) - \cosh 2\bar{k}D]. \tag{18}$$

It is also regular but attains a singular K-dV limit,

$$\bar{\varphi}_p \xrightarrow{\text{K-dV}} 2\bar{k}^2D\csc^2(\bar{k}x + \bar{\omega}t + \bar{\eta}). \tag{19}$$

In the Benjamin-Ono limit, both (14) and (18) approach constants, a trivial solution to Benjamin-Ono equation, not the Bnejamin-Ono soliton (5). On the other hand, it is interesting to note that in the limit $k \to 0$, solutions (16) and (18) are reduced to an algebraic soliton:

$$\varphi_a = -2(1/D)[(1/D)^2(x + t/D + \eta)^2 + 1]^{-1}. \tag{20}$$

This is very similar to the Benjamin-Ono soliton (5) except that here the parameter $1/D$ is fixed while the parameter v in (5) is arbitrary. It appears to us that the solutions obtained above are not general enough ro recover the Benjamin-Ono soliton. On the other hand, the above discussion certainly demonstrates that K-dV and Benjamin-Ono solitons do not come from the same parent solution of Eq. (8). They correspond to different branches of solutions. There is no smooth transition from one kind of soliton to another just from variation of the fluid depth.

To find two-soliton solutions of Eq. (8), we let[7]

$$f_2 = 1 + \exp[2(k_1 x + \omega_1 t + \eta_1)] + \exp[2(k_2 x + \omega_2 t + \eta_2)] + \exp\{2[(k_1 + k_2)x + (\omega_1 + \omega_2)t + \eta_1 + \eta_2 + A_{12}]\}. \qquad (21)$$

Substituting it into (11), we get

$$\exp(2A_{12}) = -\frac{(\omega_1 - \omega_2)\sin^2(k_1 - k_2)D - 2(k_1 - k_2)^2 \omega_2 2(k_1 - k_2)D}{(\omega_1 + \omega_2)\sin^2(k_1 + k_2)D - 2(k_1 + k_2)^2 \cos 2(k_2 + k_2)D}, \qquad (22)$$

where ω_1 and ω_2 are functions of k_1 and k_2 as given in (12). The many choices of k_1, k_2, η_1, and η_2 would then yield various kinds of solutions: a nonalgebraic two-soliton solution, a two-period solution,[8] a one-soliton, one-period solution, etc. However, the two-soliton algebraic solution does not exist in accord with the case of K-dV equation.[9] On the other hand, these solutions will tend to legitimate K-dV soliton limits but not the Benjamin-Ono soliton limit as in the case of single solitons.

In general, for N-soliton solutions, we have

$$f_N = \sum_{\mu=0,1} \prod_{i=1}^{N} \exp(\mu_i \theta_i) \prod_{i<j}^{N} \exp(2\mu_i \mu_j A_{ij}), \qquad (23)$$

where $\theta_i = 2k_i x + 2\omega_i t + 2\eta_i$, and A_{ij} are given as in (22).

In conclusion, we have demonstrated the existence of multisoliton solutions for the internal wave equation with finite depth. Although Eq. (8) tends to either the Korteweg–de Vries or the Benjamin-Ono limit as $D \to 0$ or $D \to \infty$, the soliton solutions we obtained have only the Korteweg–de Vries limit. Therefore, we do not have a smooth transition from one kind of soliton to another with varying fluid depth.

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