Pion Form Factor

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We give the normalized leading asymptotic Q^2 dependence of the pion form factor in quantum chromodynamics: $F_\pi(Q^2)\xrightarrow[Q^2\to\pm\infty]{} -2f_\pi^2/bQ^2\ln|Q^2|$, where f_π is the pion decay constant and $b=(11-\frac{2}{3}N_f)/16\pi^2$. Up to non-leading-logarithmic corrections, this is equivalent to $F_\pi(Q^2)\xrightarrow[Q^2\to\pm\infty]{} 8\pi\alpha_s(Q^2)f_\pi^2/(-Q^2)$. These results are obtained by solving the light-cone pion Bethe-Salpeter equation in quantum chromodynamics to leading-logarithmic accuracy.

In this Letter we present the solution of the quantum-chromodynamics (QCD) pion Bethe-Salpeter (BS) equation on the light cone. We normalize the resultant light-cone pion BS wave function to give the correct pion decay rate. This then enables us to give the asymptotic behavior of the pion electromagnetic form factor,

$$F_\pi(Q^2)\xrightarrow[Q^2\to\pm\infty]{} -2f_\pi^2/bQ^2\ln|Q^2|. \tag{1}$$

Here $b=(11-\frac{2}{3}N_f)/16\pi^2$ and f_π is the pion decay constant (≈ 132 MeV). To the leading-logarithmic accuracy of our calculation, this is equivalent to

$$F_\pi(Q^2)\xrightarrow[Q^2\to\pm\infty]{} 8\pi\alpha_s(Q^2)f_\pi^2/(-Q^2). \tag{2}$$

Here we sketch the derivation of these results in momentum space, some aspects of which have been previously given by Jackson.[1] Details of this calculation and aspects of the solution of the BS equation not essential to obtaining (1) will be given elsewhere.[2]

The pion BS wave function[3] can be decomposed in terms of four Lorentz-invariant functions $\varphi_i(p\cdot k,k^2)$:

$$\Phi(p,k)=\int e^{ik\cdot x}d^4x\langle 0|T[u\tfrac{1}{2}(x)\bar{d}(-\tfrac{1}{2}x)]|\pi^+\rangle$$
$$=\gamma_5\{\varphi_1\not{p}+\tfrac{1}{2}\varphi_2[\not{k},\not{p}]+\varphi_3+\varphi_4\not{k}\},$$

where p is the pion momentum and $2k$ is the relative momentum of the q and $\bar{q}$. The BS equation for Φ in terms of the two-particle irreducible kernel $K(k_1,k_2;l_1,l_2)$ is shown schematically in Fig. 1(a). Our aim here is to find $\varphi_i(p\cdot k,k^2)$ when both $p\cdot k$ and k^2 are large and spacelike, since this determines the asymptotic behavior of $F_\pi(Q^2)$. Thanks to the asymptotic freedom of QCD, when all the invariants are large and spacelike the leading behavior of the kernel is given by one-gluon exchange with a running coupling constant.[4] However, the loop integral, d^4l, involves regions for which l_1^2 and/or l_2^2 are on the order of hadron masses. Nonetheless, as long as the true kernel is not anywhere more singular than the one we use, our result is correct to leading logarithms. Since confinement presumably implies that the true quark amplitudes vanish more rapidly at large distances,[5] we can reliably calculate to leading-logarithmic accuracy using this one-gluon-exchange with running-coupling-constant approximation to the kernel.[6] Working to this accuracy we are required by consistency to replace $\alpha_s(k^2)$ in the asymptotic kernel by $(4\pi b\ln|k^2|)^{-1}$, since it is misleading to use $[4\pi b\ln(|k^2|/\Lambda^2)]^{-1}$ and identify Λ with something measured, e.g., in electroproduction: unless the next-to-leading corrections are computed, Λ has no meaning.[7]

Only the "γ_5-odd" pieces of the wave function, φ_1 and φ_4, contribute to the pion decay constant f_π {φ_2 and φ_3 contributions are multiplied by the trace of an odd number of γ matrices [Fig. 1(b)]} and make leading contributions to the large-Q^2 form factor. In the gauge in which the asymptotic kernel is

$$K^{\mu\nu}=-i[g^{\mu\nu}-(1-\lambda)l^\mu l^\nu/l^2]/bl^2\ln|l^2|,$$

they satisfy, to leading-logarithmic accuracy,

$$(-k^2\varphi_1-\tfrac{1}{2}p\cdot k\varphi_4)\not{p}+(2p\cdot k\varphi_1+k^2\varphi_4)\not{k}$$
$$=\frac{4i}{3b}\int\frac{d^4l}{(2\pi)^4l^2}\frac{1}{\ln|l^2|}\left[\left((1+\lambda)\not{p}+(1-\lambda)\frac{2p\cdot l}{l^2}\not{l}\right)\varphi_1+\left((1+\lambda)\not{k}+(3-\lambda)\not{l}+(1-\lambda)\frac{2k\cdot l}{l^2}\not{l}\right)\varphi_4\right]. \tag{3}$$

Solution of (3) is best accomplished using a spectral decomposition,[8]

$$\Phi_{1,4}(k^2, p\cdot k)=\int_{-1/2}^{1/2}\int_0^\infty \frac{g_{1,4}(\xi,t)d\xi dt}{(k^2-2\xi p\cdot k - t + i\epsilon)^3}. \tag{4}$$

Substituting (4) into (3) leads[1,2] to differential equations for the g_i with solutions of the form

$$g_i^{(m)}(\xi,t)=f_i^{(m)}(\xi)/(\ln t)^{1+\delta_m}. \tag{5}$$

The solutions only exist for particular values for δ, $\delta_m=\frac{4}{3}\{[1-2/m(m+1)]/(11-\frac{2}{3}N_f)\}-2\delta_F$, where $\delta_F=\frac{2}{3}\lambda/(11-\frac{2}{3}N_f)$ is the fermion anomalous dimension. For our application we will only need the leading solution $g_i^{(1)}$, for which $\delta_1=-2\delta_F$. The corresponding $f^{(m)}$'s are, e.g.,

$$f_1^{(1)}+\xi f_4^{(1)}\propto 1-(2\xi)^2,\quad f_1^{(3)}+\xi f_4^{(3)}\propto 1-6(2\xi)^2+5(2\xi)^4,$$
$$f_1^{(5)}+\xi f_4^{(5)}\propto 1-15(2\xi)^2+35(2\xi)^4-21(2\xi)^6, \tag{6}$$

with $f_4^{(m)}(\xi)$ given in terms of these by

$$\int_{-1/2}^{1/2}\frac{f_4^{(m)}(x)dx}{(x-\xi)^2}=\frac{1}{6\pi^2\delta_m b(\xi^2-\frac{1}{4})}\int_{-1/2}^{1/2}\frac{[f_1^{(m)}(x)+xf_4^{(m)}(x)]dx}{(\xi-x)}. \tag{7}$$

Since Eqs. (3) are homogeneous, only the relative normalizations of the f_i's are determined so far. Note that in a gauge with no fermion anomalous dimensions (where the truncated and untruncated wave functions have the same anomalous dimensions) $\delta_1=0$, as required by the partial conservation of the axial vector current, $\bar{d}\gamma^\mu\gamma^5 u$.

The dependence of f_π on the wave functions is [see Fig. 1(b)] given by

$$f_\pi=12\int\frac{d^4k}{(2\pi)^4}\left(\varphi_1+\frac{p\cdot k}{m_\pi^2}\varphi_4\right). \tag{8}$$

A little algebra shows that only the $m=1$ solution [Eq. (6)] makes a nonvanishing contribution to f_π (higher-m solutions oscillate in ξ and integrate to zero). This then gives, for the leading contribution

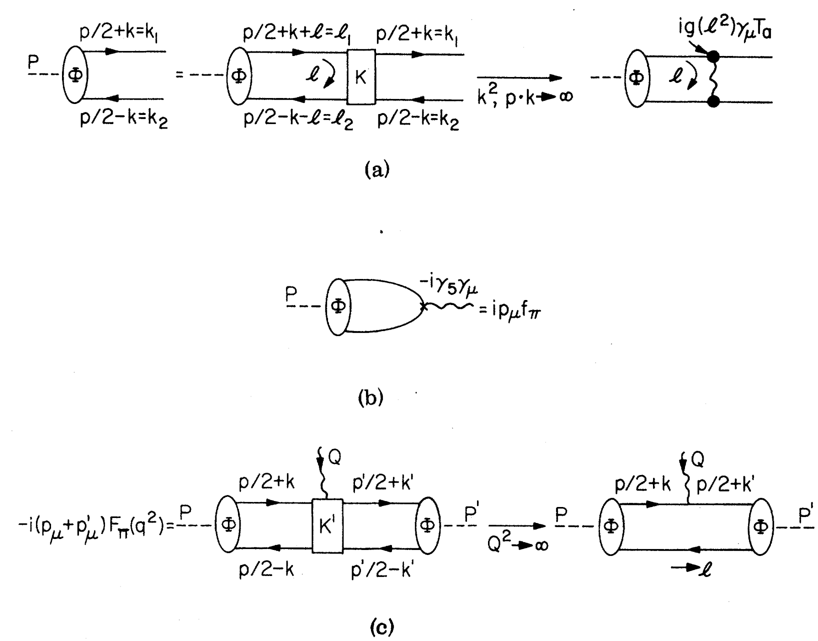

FIG. 1. (a) Pion Bethe-Salpeter equation for $\Phi(p\cdot k, k^2)$, (b) pion decay constant f_π in terms of Φ, and (c) pion form factor in terms of Φ.

to $g_1 + \xi g_4$:

$$g_1(\xi, t) + \xi g_4(\xi, t) = 4i\pi^2 \delta_1 f_\pi (1 - 4\xi^2)/(\ln t)^{1+\delta_1}. \tag{9}$$

Since the BS equation gave the relative normalization of g_1 and g_4, the leading piece of the wave function is now completely specified. δ_1 is gauge dependent and in (9) we have taken it small but positive $\to 0$ in order that all the integrals are well defined.

The new physics emerges when we calculate $F_\pi(Q^2)$ for large Q^2, as shown in Fig. 1(c). *A priori* the relevant kernel involves all two-particle irreducible gluon exchanges. However, the actual QCD wave functions (4)–(7) fall off so slowly in the (four-momenta)^2 of their legs that the larger phase space associated with large k_i^2 "wins" and calculation shows that the intermediate quark lines tend to have large k_i^2. This means that to leading-logarithmic accuracy only the lowest-order kernel is required, as shown in Fig. 1(c). Then we have

$$F_\pi(Q^2) \xrightarrow[Q^2 \to \pm\infty]{} -\frac{3i}{p \cdot p'} \int \frac{d^4 l}{(2\pi)^4} l^2 \varphi_4(k^2, p\cdot k) \cdot [2Q^2 \varphi_1{}^*(k'^2, p'\cdot k') + (Q^2 + 4p\cdot l)\varphi_4{}^*(k'^2, p'\cdot k')]. \tag{10}$$

At this point we use Eq. (4) (the higher-m solutions give nonleading contributions in Q^2) to obtain from Eq. (10) the central result, Eq. (1):

$$F_\pi(Q^2) \xrightarrow[Q^2 \to \pm\infty]{} -2f_\pi{}^2/bQ^2 \ln|Q^2|.$$

The gauge dependence, still evident in Eq. (9), cancels upon integration over t when evaluating Eq. (10) in terms of the spectral representation. The presence of the factor $f_\pi{}^2$ is not surprising, and from Born-diagram calculations the factor $1/b \sim \alpha_s(Q^2)$ is familiar. However, one might wonder why the $1/b$ factor does not disappear here into f_π [Fig. 1(c)]. It does not because of a factor $1/b$ between g_4 and $g_1 + \xi g_4$ [Eq. (7)].

We show in Ref. 2 that Eq. (1) is valid to leading-logarithmic accuracy for both spacelike and timelike Q ($Q^2 < 0$ and > 0, respectively, in our metric), in sign as well as functional dependence on $|Q^2|$. Although the sign of $F_\pi(Q^2)$ is only experimentally determined at $Q^2 = 0$, it is significant that for large spacelike Q^2 we find $F_\pi(Q^2)$ has the same sign (>0) as at $Q^2 = 0$. Furthermore, our result $F_\pi(Q^2) < 0$ for large timelike Q^2 is consistent with the negative value of $\mathrm{Re}F_\pi(Q^2)$ above the ρ resonance [asymptotically F_π is dominated by $\mathrm{Re}F_\pi$ to leading-logarithmic accuracy, and so Eq. (1) is actually for $\mathrm{Re}F_\pi(Q^2)$]. Thus there is no need for any zero in $|F_\pi(Q^2)|^2$ in the spacelike region, or in the leading asymptotic contribution to $|F_\pi(Q^2)|^2$ in the timelike region.

As stressed above, Eq. (1) is only experimentally significant when Q^2 is so large that in the expression $-2f_\pi{}^2/bQ^2 \ln(Q^2/\Lambda^2)$ it is irrelevant what choice is made for Λ. In our calculation two mechanisms are responsible for generating a scale, Λ: the renormalization of QCD (this appears in higher-order contributions in the kernels) and the infrared cutoff induced by confinement. The QCD scale parameter presumably is of the same order of magnitude as for deep inelastic scattering, whereas the infrared scale ought to be related to the pion's size. Since for the largest available $Q^2 = (3.1)^2$ GeV^2, $\ln[Q^2/(0.3)^2] = 4.7$, whereas $\ln(Q^2/1^2) = 2.2$, the natural level of error in using Eq. (2) at present Q^2 is greater than a factor of 2. Furthermore, although the neglect of the subasymptotic pieces of the wave function [$m > 1$ in Eqs. (5) and (6)] is mathematically correct for sufficiently large Q^2, for our Q^2's this may be a poor approximation since they are subasymptotic by only a fractional power of $\ln Q^2$. A means of obtaining the relative normalizations of the $f^{(m)}$'s for $m \geq 1$ would be very useful phenomenologically.[9] Nonetheless, it is relevant and interesting that Eq. (2) is accurate to better than an order of magnitude: The largest (spacelike) Q^2 data are for $Q^2 = -4$ GeV^2 and give[10] for "α_s," $-Q^2|F_\pi(Q^2)|8\pi f_\pi{}^2 = 0.9 \pm 0.2$ [and for timelike $Q^2 = (3.1\text{ GeV})^2$, $Q^2|F_\pi(Q^2)|8\pi f_\pi{}^2 = 1.6^{+0.5}_{-0.8}$ (Ref. 11)].

We close with a few remarks. Our result, Eq. (1), while not of great experimental relevance at present, is nonetheless important in that it confirms the power dependence of $F_\pi(Q^2)$, $1/Q^2$, predicted by Brodsky and Farrar,[12] while showing how QCD can be used to compute correctly the leading-logarithmic behavior of certain exclusive hadronic processes. QCD was crucial in several ways in this analysis: Without asymptotic freedom the ladder approximation would not necessarily have given leading-logarithmic accuracy. Furthermore, confinement served to *decrease* the uncertainty coming from nonperturbative infrared regions.[13] Finally and probably most importantly, if the wave functions had not had the "extra" logarithmic damping on the light cone provided by asymptotic freedom, and if the spin structure (the relation between g_1 and g_4) had been different, f_π could not in general have been used to normalize $F_\pi(Q^2)$. Thus Eq. (1) in principle provides a sensitive test of QCD.

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[1]D. R. Jackson, Ph.D. thesis, California Institute of Technology, June 1977 (unpublished).

[2]G. R. Farrar and D. R. Jackson, unpublished.

[3]This BS wave function is evidently not QCD gauge invariant since local color gauge invariance means that the definition of color can be independently changed at $\frac{1}{2}x$ and $-\frac{1}{2}x$. One could work instead with the gauge-invariant object

$$\langle 0| T[u(\tfrac{1}{2}x)\exp(i\int_{-x/2}^{x/2}A\cdot dl)\bar{d}(-\tfrac{1}{2}x)]|\pi^+\rangle$$

(where A is a matrix in color space); however, in that case one must analyze more complex kernels involving incoming and outgoing gluons. Since the final physical results are gauge independent we have chosen to employ the standard wave function, although it's anomalous dimensions are not in general gauge invariant [see Eq. (6)]. Since the two descriptions become equivalent as $x \to 0$, their leading k^2 dependence is the same.

[4]Since the pion pole is explicitly removed, no dangerous effects are expected to be induced by the continuation of $(k_1+k_2)^2=p^2\to m_\pi^2$.

[5]This is confirmed by perturbative studies of QCD [J. M. Cornwall and G. Tiktopoulos, Phys. Rev. Lett. 35, 338 (1975)] which indicate that amplitudes involving colored states are exponentially damped as those states approach "mass shell."

[6]Note, however, that obtaining nonleading logarithms in Φ and $F_\pi(q^2)$ is much more subtle: It is not sufficient to treat the kernel to higher order in $\alpha_s(k^2)$ unless it can be shown that the uncomputable contributions from the low-momentum region in the loop integral are less important.

[7]We are grateful to Douglas Ross for discussions of this point.

[8]G. C. Wick, Phys. Rev. 96, 1124 (1954). Using (4), the solution of the BS equation for spacelike variables gives φ_i for the spacelike and timelike regions.

[9]S. Brodsky and P. Le Page [reported at the Caltech QCD Workshop, February 17, 1979 (unpublished)] have obtained the same results for the leading behavior of $F_\pi(Q^2)$. Their method enables them to find easily, by working in axial gauge, the correct nonleading anomalous-dimension terms in $F_\pi(Q^2)$. In order for us to obtain the nonleading powers we would have to include higher-order corrections to the $q\bar{q}$-$q\bar{q}$ current kernel [Fig. 1(c)]. If the relative normalizations of these subasymptotic pieces can be determined, it will be very important phenomenologically. They have also informed us of a third, independent derivation of Eq. (2) by A. V. Efremov and A. V. Radyushkin, to be published.

[10]C. J. Bebek *et al.*, Phys. Rev. D 13, 25 (1976).

[11]W. Braunschweig *et al.*, Phys. Rev. Lett. B63, 487 (1976).

[12]S. J. Brodsky and G. R. Farrar, Phys. Rev. Lett. 31, 1153 (1973), and Phys. Rev. D 11, 1309 (1975).

[13]It has been known for a long time that in many model regions of the wave function other than short distance may make important contributions to $F_\pi(Q^2)$ even for $Q^2 \to \infty$ (cf., e.g., Ref. 4). The reasons that this is not the case for QCD were outlined briefly in the text and will be fully developed in Ref. 2. Other references relating to this question and to the issue of the form of the result are C. G. Callen and D. J. Gross, Phys. Rev. D 11, 2905 (1975); P. Menotti, Phys. Rev. D 13, 1778 (1976); M. L. Goldberger, A. H. Guth, and D. E. Soper, Phys. Rev. D 14, 1117 (1976). Emphasizing an operator approach to the problem is A. M. Polyakov, in *Proceedings of the International Symposium on Lepton and Photon Interactions at High Energy, Stanford, California, 1975*, edited by W. T. Kirk (Stanford Linear Accelerator Center, Cal., 1975); V. L. Cherniyak, A. I. Vainshtein, and V. I. Zakharov, private communication.