drawn to the book by G. Z. Gershuni and E. M. Zhukhovitskii, *Convective Stability of Incompressible Fluids* (Keter Publishing House, Jerusalem, 1976); we found that in Sects. 3, 5, and 6 of this monograph, the authors discuss the theoretical aspects of the same problem as analyzed in our present work.

[12]Note that the corresponding Brunt-Väisälä frequency, for purely gravitational waves, defined as $N = (-\nu\kappa R d^{-4})^{1/2}$ (see Ref. 2), would then have the value $6.61\ \mathrm{sec}^{-1}$.

[13]Without a considerably more sophisticated detection design, it is impossible to predict reliably the relative weight of the two signals. Note that the same cause may be the origin of similar problems in some earlier work (see Ref. 8).

[14]The Fourier analysis along the horizontal direction ($\sim$ wavelength l) could also be replaced by a real-space analysis. This method was used in the case of unstable stratification to study the response of the system to a local temperature excitation [B. M. Berkovsky *et al.*, J. Fluid Mech. 89, 173 (1978)] and could be applied to the present problem.

Stochastic Heating of a Large-Amplitude Standing Wave

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Stochastic behavior of particle motions in an electrostatic standing wave is analyzed; its significance in supplementary heating of plasmas is discussed.

In heating plasmas with high-power rf waves, a symmetrical structure[1,2] is often used, which leads to large-amplitude standing waves in the plasma. As a result, stochastic heating of low-energy particles can readily occur. If this stochastic heating happens at the surface of a tokamak where energy is poorly confined, a significant fraction of the rf energy may be lost. The onset of stochasticity may be visualized by drawing phase-space contours of constant particle energies in the presence of each of the two oppositely directed traveling waves alone. If the wave amplitude is sufficiently large, the overlapping of contours allows particles to execute random walks in these two waves. Therefore, the constant of the motion disappears and the trajectory appears to be chaotic.

Applications of stochastic acceleration[3] have been studied by Smith and Kaufman[4] for an obliquely propagating wave and by Kearney and Bers[5] for a perpendicularly propagating wave. Here, we consider particle motions along the magnetic field lines, or in an unmagnetized plasma in the presence of a standing electrostatic wave, and discuss the stochasticity boundary, the scaling laws of heating, and the modification of the plasma dielectric function due to stochastic electron motion.

We begin our analysis by writing the equation of motion for a particle in a standing wave of frequency ω_0 and wavelength $\lambda_0 = 2\pi/k_0$:

$$d^2X/dT^2 = p \sin X \sin T, \tag{1}$$

where in dimensionless units, $p = eE_0 k_0/m\omega_0^2$ is the electric field strength, and $X = k_0 x$, $T = \omega_0 t$ are the spatial and temporal coordinates. The onset of stochasticity may be found by the criterion of overlapping resonances.[3] Decomposing the standing wave into two traveling waves and considering the particle orbits in each of the traveling waves, one finds the particle trapping regions overlap when $p = 1$. This gives a stochasticity boundary at $p \sim 1$. The physical origin of the stochasticity may be illustrated from the spectrum $X_\omega = (1/2\pi)\int_{-\infty}^{+\infty} X(T) e^{i\omega T} dT$. To study the character of the spectrum, we linearize Eq. (1) around the fixed points $X_0^n = n\pi$ (n is an integer) to give

$$d^2X/dT^2 = pX \sin T, \tag{2}$$

where $(-1)^n$ is absorbed into $\sin T$. Equation (2) is a special case of Mathieu's equation, $d^2\psi/d\varphi^2 = (b - h^2\cos^2\varphi)\psi$, with the relation $b = h^2/2 = 4p$. It is a general property of Mathieu's equation to exhibit alternating regions of bounded and unbounded solutions in (b, h) space. When $p > p_c = 0.456$, the solution is unbounded.[6] To understand this "stochastic" instability, Eq. (2) is solved by Fourier analysis,

$$\omega^2 X_\omega + (p/2i)(X_{\omega+1} - X_{\omega-1}) = 0. \tag{3}$$

Substituting $\omega \pm 1$ for ω and combining the equations, we obtain

$$\epsilon_{st}(\omega)X_\omega \equiv \left\{\omega^2 - \frac{p^2}{4}\left[\frac{1}{(\omega-1)^2} + \frac{1}{(\omega+1)^2}\right]\right\}X_\omega = \frac{p^2}{4}\left[-\frac{X_{\omega-2}}{(\omega-1)^2} - \frac{X_{\omega+2}}{(\omega+1)^2}\right]. \tag{4}$$

Since the onset of the stochastic instability is expected to occur when $p < 1$, justified *a posteriori*, we may neglect the term on the right-hand side of Eq. (4) which is shown to be of order p^4 by substituting $\omega \pm 2$ for ω in Eq. (4) and combining the resulting equations. We note that the dispersion relation obtained resembles that of the two-stream instability and can be solved analytically by making it into a cubic equation in $\Omega = \omega^2$. Marginal stability occurs when two of the roots of Ω are equal, which gives $p_c = 0.473$. This value is slightly higher than that obtained from Mathieu's equation, since higher harmonics were neglected and the harmonic-generation process could have contributed to the onset of the instability. Six modes are present in $\epsilon_{st}(\omega) = 0$: two sidebands around each of the two traveling waves, and two "stochastic" modes with real parts of frequencies at $\omega_{st} \simeq \pm p/\sqrt{2}$. Saturated fluctuation spectrum at the stochastic modes for $1 > p > p_c$ is expected for the nonlinear system, Eq. (1). Numerical results are shown in Fig. 1. When p is small, particle motions are characterized by a shifted pump-wave frequency with tiny contributions from sidebands. For $1 > p > p_c$, X_ω develops, with distinguished peaks at the stochastic modes, a broadband noise indicating a possible long limit-cycle period. The spectrum resembles a fully ergodic system which is expected to yield simply white noise. The particle trajectory is also analyzed by the stroboscopic method.[7] On a stroboscopic plot, a regular trajectory due to the constraint of the constant of motion may appear as a set of points to lie on one or more curves, while a stochastic trajectory appears as a set of scattered points. With the development of broadband noise for $p > p_c$, it is expected that no simple pattern of curves would result [Fig. 1(c)]. It appears therefore more accurate to regard p_c as the stochasticity boundary.

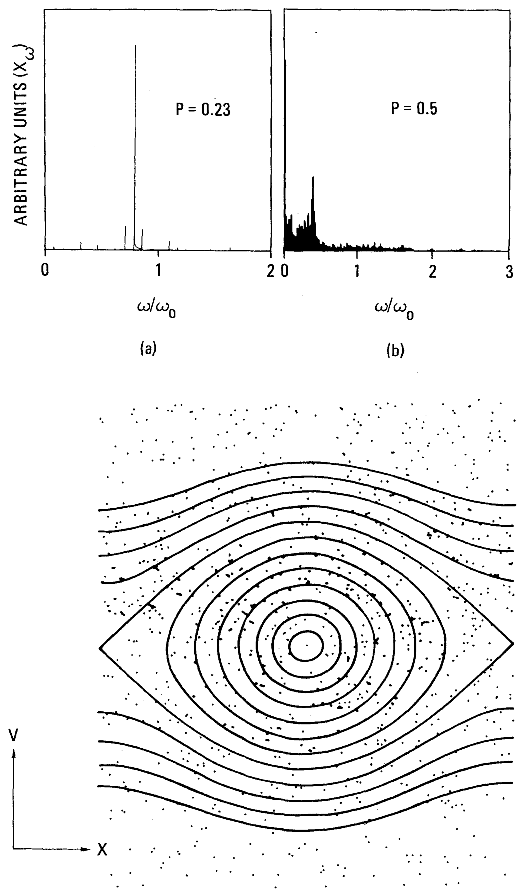

FIG. 1. (a), (b) X_ω vs ω. (c) Stroboscopic plot of $\ddot{X} = p \sin X \sin T$, $p = 0.8$, initial position at 0 point, one dot per wave period. Solid lines are the regular trajectories for $\ddot{X} = p \sin X$ at different energies.

Heating of particles, an irreversible process, arises because the stochasticity leads to irreversibility through the mixing process: A small element of the phase-space fluid develops into a filamentary structure throughout the accessible phase space. Irreversibility may also be accomplished in reality if the period of the limit cycle exceeds, for example, the collisional time, or if the boundary effects of a finite-size antenna are included. To study the scaling laws of heating, we calculate the energy gain for both large and small p values. For $p \ll 1$, the averaged kinetic energy is calculated iteratively, $\langle V^2 \rangle = V_0^2 + (p^2/4)(V_0^2 + 3)/(V_0^2 - 1)^2$, where V_0 is the initial velocity and V_0 and V are normalized to the phase speed. The energy gain is proportional to p^2. The energy absorption possible from two Landau resonances is excluded in order to compare with stochastic heating in which Landau damping plays no part. When $p \gg 1$ and the particle motion is stochastic, excluding Landau damping, the distribution function satisfies

$$\frac{\partial f}{\partial T}+V\frac{\partial f}{\partial X}+p\sin X\sin T\frac{\partial f}{\partial V}=0. \tag{5}$$

Since $p=\omega_b^2/\omega_0^2$, where $\omega_b=(eE_0k_0/m)^{1/2}$ is the bounce frequency, Eq. (5) may be studied by multiple-time expansions. If we define $\tau=p^{1/2}T$, $v=V/p^{1/2}$, $x=X$, and $H=v^2/2+\cos x\sin\epsilon\tau$, and change variables from (X,V,T) to (x,H,τ), Eq. (5) is transformed into

$$\frac{\partial f}{\partial \tau}+v(x,H,\tau)\frac{\partial f}{\partial x}+\epsilon\cos x\cos\epsilon\tau\frac{\partial f}{\partial H}=0. \tag{6}$$

Here, $\epsilon=1/p^{1/2}\ll 1$. If one assumes

$$f=f^{(0)}+\epsilon f^{(1)}+\epsilon^2 f^{(2)}+\ldots,$$

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial\tau_0}+\epsilon\frac{\partial}{\partial\tau_1}+\epsilon^2\frac{\partial}{\partial\tau_2}+\ldots,$$

the lowest-order equation is

$$\frac{\partial f^{(0)}}{\partial\tau_0}+v\frac{\partial f^{(0)}}{\partial x}=0. \tag{7}$$

The physical solution of interest is the temporal evolution of the energy; we choose $f^{(0)}=f^{(0)}(H,\tau_1,\tau_2,\ldots)$. To next order,

$$\frac{\partial f^{(0)}}{\partial\tau_1}+\frac{\partial f^{(1)}}{\partial\tau_0}+v\frac{\partial f^{(1)}}{\partial x}+\cos x\cos\tau_1\frac{\partial f^{(0)}}{\partial H}=0. \tag{8}$$

We must eliminate the secularity from Eq. (8). The first term independent of τ_0 and x is a secular term. Since the last term averaged over the particle trajectory is nonzero, we have to let

$$\frac{\partial f^{(0)}}{\partial\tau_1}+\langle\cos x\rangle\cos\tau_1\frac{\partial f^{(0)}}{\partial H}=0, \tag{9}$$

where $\langle\cos x\rangle=(\oint\cos x\,dx/v)/(\oint dx/v)$. The integral path is taken along the particle trajectory given by $v=\pm[2(H-\cos x\sin\tau_1)]^{1/2}$. Since $\langle\cos x\rangle$ is a function of $\alpha=\sin\tau_1/H$ alone and can be written in terms of elliptic functions, the general solution to Eq. (9) is given by $f^{(0)}=f^{(0)}[H/\exp A(\alpha)]$, where $A(\alpha)=\int_0^\alpha\langle\cos x\rangle d\alpha'$. It is instructive to find the similarity solution by substituting $\tau_1=T$, $H=V^2/2p+\cos X\sin T$ back. When $V^2=pv^2$, p is scaled out of the solution. The average energy perparticle thus scales with p for $p\gg 1$.

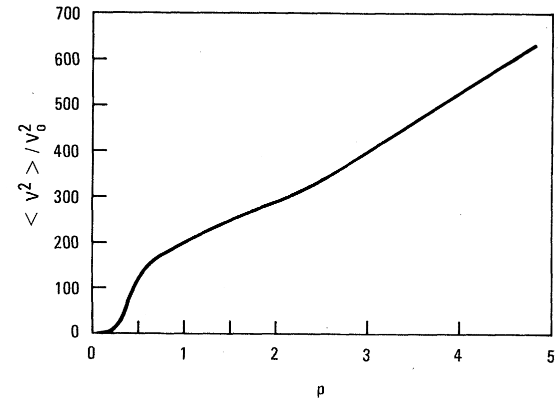

FIG. 2. $\langle V^2\rangle$ vs p. Initial temperature 10 eV, $\omega_0/k_0=1.5\times10^9$ cm/sec, $V_0=0.127$. As an example, final temperature for $p=0.7$ is around 1.4 keV.

A one-dimensional particle-simulation code was developed to test the scaling laws. The code uses 512 particles with the initial distribution Maxwellian and spatially uniform. Comparisons are made for sudden turn-on and adiabatic turn-on of the pump wave; no difference in the final stage is detected. Within a few wave periods, the averaged particle kinetic energy has reached an equilibrium value with appreciable fluctuations. Figure 2 shows time-asymptotic averaged kinetic energy versus p; qualitative agreement with the theory is obvious. The stochasticity boundary p_c appears to coincide with the inflection point of the curve.

To study the modification of the dielectric function due to stochastic electrons, we follow closely the work by Kruer, Dawson, and Sudan.[8] Equation (2) perturbed by an electric field is

$$\frac{d^2X^n}{dT^2}=(-1)^n pX^n\sin T+\frac{1}{(2\pi)^2}\int E(k',\omega')\exp(ik'X^n-i\omega'T)\,dk'\,d\omega'. \tag{10}$$

The perturbed electric field $E(k',\omega')$ is influential around the equilibrium points $X_0^n=n\pi$ and is negligible otherwise. We restrict ourselves to $p<1$ so that an expansion in p is possible while intrinsic stochasticity may still occur. Fourier analyzing Eq. (10) and retaining accuracy to order p^2, following the procedures in Ref. 6, we obtain the nonlinear electron density due to particle stochasticity. Treating the background plasma as a continuous medium with a general dielectric function $\epsilon_L(k,\omega)$, we obtain

$$\epsilon_L(k,\omega)E(k,\omega)=\frac{\Omega_{st}^2}{\epsilon_{st}(\omega)}\sum_m\left\{E(k+m,\omega)+\frac{p}{2i}\left[\frac{E(k+m,\omega+1)}{(\omega+1)^2}-\frac{E(k+m,\omega-1)}{(\omega-1)^2}\right]\right\}. \tag{11}$$

Here, $\Omega_{st}^2 = 4\pi N_{st} e^2/m_e \omega_0^2$ is the plasma frequency of the stochastic electrons, N_{st} is the number of stochastic electrons involved, and the summation is over integers ranging from $-\infty$ to $+\infty$. To allow a perturbative treatment, N_{st}/N_0 is assumed to be small, and N_0 is the averaged density. The coupling of the standing wave with one natural mode in the background plasma can then be studied.

We may take $\epsilon_L(k,\omega) = [(1-\omega_e^*/\omega)\Lambda + k_\perp^2\lambda_s^2]/(k_\perp^2+k^2)/\lambda_{De}^2$, the drift-wave dispersion relation. Here, the ion thermal effect is neglected, ω_e^* is the electron diamagnetic drift frequency, λ_s is the ion gyroradius at the electron temperature, λ_{De} is the electron Debye length, $\Lambda = 1 + i\delta$, and δ represents the dissipative effect; all quantities are normalized to ω_0 and k_0. Since $\epsilon_{st}(\omega)$ is off resonance for $\omega \sim \omega_e^* \ll \omega_{st} < 1$, Eq. (11) is simplified by keeping the largest nonlinear term:

$$\left(1 - \frac{\omega_e^*}{\omega}\right)\Lambda + k_\perp^2\lambda_s^2 \approx \frac{k^2\lambda_{De}^2\Omega_{st}^2}{\omega_e^{*2} - p^2/2}. \tag{12}$$

Destabilization of drift waves because of stochastic electrons arises if $\omega_e^* < p/\sqrt{2}$.

These results can be compared with some experimental observations. Hooke and Bernabei[1] reported that the penetration of rf energy is a strong function of pump-wave frequency where standing waves are generated from plates. The electric field E in the experiment is believed to exceed 5 V/cm, $\lambda_0 \sim 20$ cm. The parameter p found from $n_\parallel^2 e\varphi/m_e c^2$ decreases with frequency from 7 at 15 MHz to 1 at 40 MHZ. The rf energy penetration increases with frequency which was explained (Ref. 1) by the linear ray trajectory $V_g = V_{ph}(1-\omega_{LH}^2/\omega^2)$ but with a discrepancy between the observed and predicted density by as much as a factor of 2–8. It is apparent from our estimate of p that stochastic heating should occur at low frequencies in that experiment and with the same frequency dependence. Drift-wave satellites were excited without a threshold which agrees with the conclusions following Eq. (12) since $\omega_e^* \ll 1$. For rf heating experiments in the dc octopole[2] with the estimated $E \sim 10$ V/cm, $\lambda_0 \sim 2$ cm, and ω_0 from 10 MHz to 1 GHz, p values vary by four orders of magnitude and may well exceed 1. Although plasma-wave coupling is excellent in the low-frequency regime, the power transmitted to the plasma core saturates when the applied power increases. These observations can be understood from the stochastic heating mechanism. In other recent lower-hybrid heating experiments on Doublet IIA[9] and Alcator[10] p values might vary from 0.1 to 1; the present mechanism could be significant.

Finally, we note that a standing wave may stochastically heat electrons along the field lines or ions perpendicularly without any resonance condition. The standing wave may be created in the plasma core by launching two oppostiely directed traveling waves from the plasma edge. Accessibility of the traveling waves could easily be provided since a wide range of frequencies and wavelengths might be used.

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