Apparent Violation of Unitarity in Elastic $\pi^- p$ Scattering

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We deduce that if $f(\cos\theta)$ has nonnegative partial waves, then

$$[f(\cos\theta)/f(1)]^2 \leq \tfrac{1}{3}+\tfrac{2}{3}f(\tfrac{1}{2}(3\cos^2\theta-1))/f(1)$$

for $-1\leq\cos\theta\leq 1$. Applying this, we show that recent determinations of the $\pi^- p$ slope parameter for $|t|\leq 0.04$ $(\mathrm{GeV}/c)^2$ at 100 GeV/c at CERN, together with those at larger $|t|$ at Fermilab, imply a substantial violation of unitarity, though each determination by itself is consistent with unitarity.

A recent CERN Super Proton Synchrotron experiment[1] on $\pi^- p$ elastic scattering measuring both the proton recoil and the forward-scattered beam particle has reported the discovery that the slope parameter $b(t)$ in the $|t|$ range 0.002 to 0.04 $(\mathrm{GeV}/c)^2$ is significantly higher than what was found earlier in the range 0.0375 to 0.75 $(\mathrm{GeV}/c)^2$ by Ayres *et al.*[2] For example, at 100 GeV/c the above experiments give, respectively,

$$b(t)=11.3\pm 0.3\ (\mathrm{GeV}/c)^{-2} \text{ for } 0.002\leq|t|\leq 0.04\ (\mathrm{GeV}/c)^2 \tag{1}$$

and

$$b(t)=9.2\pm 0.1\ (\mathrm{GeV}/c)^{-2}-2|t|\times(2.4\pm 0.2) \text{ for } 0.0375\leq|t|\leq 0.8\ (\mathrm{GeV}/c)^2, \tag{2}$$

suggesting a break in slope at $|t|\sim 0.04$ $(\mathrm{GeV}/c)^2$. Further, Ref. 1 gives for the ratio ρ of real to imaginary part at 100 GeV/c, for $|t|<0.04$ $(\mathrm{GeV}/c)^2$,

$$\rho=+0.023\pm 0.013, \tag{3}$$

in agreement with dispersion relations. Exploiting an inequality on functions with nonnegative partial waves (in particular, the spin-nonflip absorptive part), and assuming qualitative validity of (3), we seek to check the consistency of the results (1) and (2). We find that the break in slope implied by simultaneous validity of these results would require a substantial violation of unitarity.

(1) *Inequalities on functions of positive type on the three-dimensional rotation group.*—Such functions $f(\hat{n}\cdot\hat{n}')$ are characterized by a partial-wave expansion with nonnegative coefficients,

$$f(\hat{n}\cdot\hat{n}')\equiv f(\cos\theta)=\sum_{L=0}^{\infty}(2L+1)f_L P_L(\cos\theta),\quad f_L\geq 0, \tag{4}$$

and hence obey

$$\int d\Omega(\hat{n}')\int d\Omega(\hat{n})\psi^*(\hat{n}')f(\hat{n}\cdot\hat{n}')\psi(\hat{n})=\sum_L 4\pi f_L\sum_{m=-L}^{L}\left|\int d\Omega(\hat{n})\psi(\hat{n})Y_{Lm}(\hat{n})\right|^2\geq 0, \tag{5}$$

where $\psi(\hat{n})$ is any complex function. In particular, the $N\times N$ matrix whose elements are

$$f_{ij}\equiv f(\hat{n}_i\cdot\hat{n}_j),\quad i,j=1,2,3,\ldots,N, \tag{6}$$

must have nonnegative eigenvalues. If we choose $\hat{n}_2,\hat{n}_3,\hat{n}_4$ to be symmetrically placed on a cone of half-angle θ around $\hat{n}_1$ which is chosen as the z axis, i.e.,

$$\hat{n}_{i+2}=(\sin\theta\cos(2\pi i/3),\ \sin\theta\sin(2\pi i/3),\ \cos\theta),$$
$$i=0,1,2, \tag{7}$$

then the nonnegativity of eigenvalues of the corresponding 4×4 matrix requires that

$$[f(\cos\theta)/f(1)]^2\leq\tfrac{1}{3}+\tfrac{2}{3}f(\tfrac{1}{2}(3\cos^2\theta-1))/f(1),\quad \text{for } -1\leq\cos\theta\leq 1. \tag{8}$$

Hence, if $[f(\cos\theta)/f(1)]^2>\frac{1}{3}$, then $f(\frac{1}{2}(3\cos^2\theta-1))>0$. Further,

$$f(-\tfrac{1}{3})\geq-\tfrac{1}{3}f(1). \tag{9}$$

Similarly if we choose $\vec{n}_1 \cdot \vec{n}_2 = \vec{n}_1 \cdot \vec{n}_3 = \cos\theta$, and $\cos(2\theta) \le \vec{n}_2 \cdot \vec{n}_3 \le 1$, the nonnegativity of eigenvalues of the corresponding 3×3 matrix requires that

$$[f(\cos\theta)/f(1)]^2 \le \tfrac{1}{2} + \tfrac{1}{2} f(\cos\theta')/f(1), \quad \text{for } \cos(2\theta) \le \cos\theta' \le 1. \tag{10}$$

The interest of the elementary inequalities (8) and (10) is their immediate applicability to many physical problems as nonlinear conditions due to unitarity. For example, we may obtain nonlinear inequalities on pion-pion amplitudes inside the Mandelstam triangle which we shall discuss elsewhere. Here we present restrictions on elastic amplitudes in the physical region.

We now examine elastic absorptive cross sections and slope parameters. The imaginary part of the elastic-scattering amplitude $f(s,\cos\theta)$ for two spinless particles has nonnegative partial waves in the physical region. Consequently inequalities (8) and (10) apply if we choose $f(\cos\theta) = \mathrm{Im} f(s, \cos\theta)$, or $f(\cos\theta) = d^n \mathrm{Im} f(s, \cos\theta)/d(\cos\theta)^n$, $n = 1, 2, \ldots$. For the first two choices, we obtain from (8)

$$R(s, 3t(1 + t/4k^2)) \ge \{\tfrac{1}{2}[3R(s, t) - 1]\}^2 \theta(3R(s, t) - 1), \tag{11}$$

for

$$R(s, t) \equiv \frac{d\sigma^A(s, t)/dt}{d\sigma^A(s, 0)/dt}, \quad \text{and for} \quad R(s, t) \equiv \frac{d\sigma^A(s, t)/dt}{d\sigma^A(s, 0)/dt} \left(\frac{b^A(s, t)}{b^A(s, 0)}\right)^2. \tag{12}$$

Here k = c.m. momentum, $\theta(x) \equiv 1$ for $x \ge 0$, $\theta(x) \equiv 0$ for $x < 0$, $d\sigma^A/dt$ denotes the contribution of the absorptive part to the differential cross section, and b^A denotes its logarithmic derivative with respect to t.

For elastic scattering of two arbitrary-spin particles, the Mahoux-Cornille-Martin result that $d\sigma/dt^A$ has nonnegative partial waves[3] enables us to identify $f(\cos\theta)$ with $d\sigma^A/dt$ or its derivatives with respect to t, and obtain analogous inequalities.

(2) *Pion-nucleon scattering.*—The partial-wave expansion[4]

$$(f_1 + f_2 \cos\theta) = \sum_{L=0}^{\infty} [L f_{L-} + (L+1) f_{L+}] P_L(\cos\theta) \tag{13}$$

and the unitarity conditions

$$\mathrm{Im} f_{L-} \ge 0 \text{ and } \mathrm{Im} f_{L+} \ge 0 \tag{14}$$

valid in the physical region guarantee the rigorous validity of inequalities (8)–(10) with the choice $f(\cos\theta) = \mathrm{Im}(f_1 + f_2\cos\theta)$, or $d^n \mathrm{Im}(f_1 + f_2 \cos\theta)/d(\cos\theta)^n$. For comparison with experiment we make the approximation

$$\begin{aligned} d\sigma/d\Omega &= |F_{++}\cos(\tfrac{1}{2}\theta) + F_{+-}\sin(\tfrac{1}{2}\theta)|^2 + |F_{++}\sin(\tfrac{1}{2}\theta) - F_{+-}\cos(\tfrac{1}{2}\theta)|^2 \\ &= |f_1 + f_2\cos\theta|^2 + |f_2|^2 \sin^2\theta \approx (1 + \rho^2)[\mathrm{Im}(f_1 + f_2\cos\theta)]^2, \end{aligned} \tag{15}$$

and the *assumption* (as in Ref. 1) that ρ is independent of $\cos\theta$. (There is no need to neglect ρ.) We deduce, in particular,

$$|b(t)| \left(\frac{d\sigma}{dt}\bigg|_t\right)^{1/2} \ge \frac{b(0)}{2} \left(\frac{d\sigma}{dt}\bigg|_0\right)^{1/2} \left(3 \frac{d\sigma/dt|_{t_1}}{d\sigma/dt|_0} \frac{b^2(t_1)}{b^2(0)} - 1\right), \tag{16}$$

where

$$t \equiv 3t_1(1 + t_1/4k^2); \tag{17}$$

the square roots are defined to be positive, and we suppress the common label s of $b(s, t)$ and $d\sigma(s, t)/dt$. Inserting the fit to measurements at 100 GeV/c quoted in Ref. 2 into the left-hand side, and that of Ref. 1 into the right-hand side of (16), we obtain Table I. This table shows only statistical uncertainties, and reveals a substantial contradiction with unitarity. Actually Ref. 2 quotes additional systematic errors in absolute normalization (~3%) and slopes (~1.5%). If we trust the absolute normalization of Ref. 1 which quotes smaller systematic errors and matches with σ_{tot} data,[5] we might multiply the cross section of Ref. 2 by 0.946 to agree with Ref. 1 at $|t| = 0.04$; then the numbers of column 2 of

TABLE I. Comparison of inequality (16) with $\pi^- p \to \pi^- p$ experiments at 100 GeV/c, taking the lect-hand side from Ref. 2 and right-hand side from Ref. 1. A substantial violation of unitarity is evident. Changing the absolute normalization of Ref. 2 to agree with Ref. 1 at $|t| = 0.04$ (GeV/c)^2 would lead to multiplying the second column of the table by 0.973, and hence to increased contradiction.

$(-t)$ [(GeV/c)^2]	$\lvert b(t)\rvert (d\sigma/dt)^{1/2}$ (data of Ref. 2) [mb$^{1/2}$(GeV/c)$^{-3}$]	Lower bound on $\lvert b(t)\rvert (d\sigma/dt)^{1/2}$ [right-hand side of Eq. (16)] (data of Ref. 1) [mb$^{1/2}$ (GeV/c)$^{-3}$]
0.045	39.1 ± 0.7	47.0 ± 1.0
0.060	36.3 ± 0.6	42.7 ± 0.8
0.075	33.6 ± 0.6	38.7 ± 0.6
0.090	31.2 ± 0.5	34.9 ± 0.4
0.105	29.0 ± 0.5	31.3 ± 0.3

Table I would get multiplied by 0.973 making the contradiction still worse. For example, at $|t| = 0.06$, we would get (instead of $36.3 \pm 0.6 \geq 42.7 \pm 0.8$),

$$35.3 \pm 0.6 \geq 42.7 \pm 0.8 \text{ mb}^{1/2} (\text{GeV}/c)^{-3}. \tag{18}$$

Faced with this contradiction, we examine the approximation (15). The fractional errors in $d\sigma/d\Omega$ and $d(d\sigma/d\Omega)/dt$ due to it are

$$\left|\frac{f_2}{f_1 + f_2 \cos\theta}\right|^2 \sin^2\theta \tag{19a}$$

and

$$\left|\frac{f_2}{f_1 + f_2 \cos\theta}\right|^2 \frac{1}{k^2 b(t)}, \tag{19b}$$

both of which are $< 1/400$ at 100 GeV/c, and $|t| < 0.12$ (GeV/c)^2 if we assume $|f_2/(f_1 + f_2 \cos\theta)| < 1$ and $b(t) > 8(\text{GeV}/c)^{-2}$. The hypothesis of s-channel helicity conservation for the Pomeron[6] gives

$$F_{+-}/F_{++} \approx m_N(-t)^{1/2}/s,$$

i.e.,

$$|f_2/(f_1 + f_2 \cos\theta)| \approx \tfrac{1}{2}.$$

Further, amplitude analysis yields,[7] at $|t| = 0.1$ (GeV/c)^2,

$$|F_{+-}{}^0/F_{++}{}^0| (s_0/-t)^{1/2} = 0.14 \pm 0.17 \text{ and } 0.17 \pm 0.29 \tag{20}$$

at 6 GeV/c and 16 GeV/c, respectively, and at $|t| = 0.12$ (GeV/c)^2 and 6 GeV/c,

$$|F_{+-}{}^1/F_{++}{}^0| (s_0/-t)^{1/2} = 0.29 \pm 0.01. \tag{21}$$

Here $s_0 = 1$ GeV^2 and the superscripts 0 and 1 denote t-channel isospin. If the ratios in (20) and (21) are assumed not to rise significantly up to 100 GeV/c (the latter ratio is expected to vanish for $s \to \infty$), the assumption $|f_2/(f_1 + f_2 \cos\theta)| < 1$ would seem safe, even without s-channel helicity conservation.

Finally, the fractional error in $d(d\sigma/d\Omega)/dt$ due to neglecting $d\rho/dt$ (here and in Ref. 1) is $2\rho(d\rho/dt)/(1+\rho^2)b(t)$. If we assume that at $|t| = 0.04$ (GeV/c)^2, $|t\, d\rho/dt| < \rho$, i.e., the measurement of ρ in Ref. 1 has some validity, then the above fractional error is $< 3/1000$.

It appears that removal of our approximation (15) might only yield negligible corrections to the inequality (16). Its substantial violation is exemplified by Table I. The rise in slope at small $|t|$ (reported by Ref. 1) which causes this violation was also indicated by an earlier experiment[8] with larger errors. If we accept both unitarity and the results of Eqs. (1) and (2) as correct, there is room to speculate on existence of new physics in the Coulomb interference region. A new experiment for $|t| \leq 0.2$ (GeV/c)^2 will be useful in resolving or confirming the contradiction reported here.

I am grateful to G. Auberson for showing me a method of proof due to V. Glaser of the inequality $P_L(\cos\theta) \geq -\frac{1}{2}$ for $1 \geq \cos\theta \geq -\frac{1}{2}$. The method of Sec. 2 is essentially the same. I am also grateful to Virendra Singh for stimulating speculations concerning possible strong long-range forces, to D. P. Roy for guidance on status of s-channel helicity conservation and painstaking suggestions on this manuscript, and to André Martin for bringing this work to the attention of the authors of Ref. 1.

Note added.—Grafström and Ekelöf (of Ref. 1) have communicated the following ingenious suggestion to save unitarity. Instead of using Eqs. (1) and (2), use $b(-0.02)=11.3$, and $b(-0.06)$ as given by Eq. (2); then the unitarity inequality (16) yields $b(0)>12$. This suggests for $|t|\leq 0.04$, $b(t)=12-35\,|t|$, which is apparently compatible with measurements of Ref. 1. Such a strong t dependence of the slope parameter in the Coulomb interference region has never been visualized before, and, if experimentally confirmed, may have important theoretical implications.

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Equipartitioned Jets: New Tests of Quantum Chromodynamics

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We propose a modified definition of hadronic jets in quantum chromodynamics which is more selective than the Sterman-Weinberg criterion. It prohibits any two jet streams from having appreciably different energies. Our more restrictive definition provides new tests of quantum chromodynamics in high-energy experiments. We illustrate this idea for two-"quark"-jet configurations in $e^+e^-\to$ hadrons and for two-"gluon"-jet production. The necessity of exponentiating perturbative jet predictions is also discussed.

Sterman and Weinberg[1] (henceforth referred to as S-W) have argued that the appearance of hadronic jets[2] (narrow energetic conical streams of hadrons) in high-energy processes can be easily understood within the framework of perturbative quantum chromodynamics[3] (QCD). They suggest a scenario in which jetlike configurations of quarks and gluons are established at short distances and then materialize as final-state jets of physical hadrons. That is, the dynamics of quark-gluon hadronization does not disturb the initial jet's kinematic structure—a plausible assumption for very high-energy phenomena, if QCD is indeed correct.

S-W further argued that for very high-energy processes, where the effective coupling is small, QCD predictions for cross sections which are free of infrared and quark-mass singularities can be reliably calculated by use of perturbative Feynman-diagram techniques with final-state quarks and gluons. One such calculable cross section is that of jet production. For the example they considered, e^+e^- annihilation into hadrons, S-W defined a two-jet event by the following criterion: A two-jet event for $e^+e^-\to$ hadrons is one in which all but at most a fraction $\epsilon\ll 1$ of the total e^+e^- energy E is emitted within a pair of oppositely directed cones (jets) of opening half-angle $\delta\ll 1$ (i.e., the total jet energy must satisfy $E-\epsilon E\leq E_1+E_2\leq E$, where E_i is the energy of the ith jet).

Employing this definition, they found that the cross section for jet production (via $e^+e^-\to$ quark + antiquark $\to$ 2 hadron jets) into two fixed cones of solid angle Ω (with $\pi\delta^2\ll\Omega\ll 1$) at an angle θ to the e^+e^- beam line is given (to order α_s/π) by

$$\sigma(E,\theta,\Omega,\epsilon,\delta)=\left(\frac{d\sigma}{d\Omega}\right)_0\Omega\left\{1-\frac{4}{3}\frac{\alpha_s(E)}{\pi}\left[\ln\delta(4\ln 2\epsilon+3)+\frac{\pi^2}{3}-\frac{5}{2}\right]\right\},\tag{1}$$