Energy-Momentum in General Relativity

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For isolated gravitating systems of physical interest, the difference between the Arnowitt-Deser-Misner four-momentum and the Bondi four-momentum associated with a retarded instant of time is shown to equal the four-momentum carried away by the gravitational radiation emitted between infinite past and the given retarded instant.

For about fifteen years, there have been available in general relativity two distinct notions of energy-momentum for isolated systems: the Arnowitt-Deser-Misner[1] (ADM) four-momentum and the Bondi[2,3] four-momentum. The first of these quantities is a fixed four-vector having the interpretation of the *total* energy-momentum, including all contributions from the gravitational field itself, while the second is a four-vector associated with each retarded instant of time and represents the four-momentum "leftover" at that retarded instant after allowing for gravitational radiation. That these interpretations are correct has been a basic assumption underlying a variety of investigations in general relativity. For example, the theoretical prediction that gravitational waves do carry away energy-momentum, and that the energy so carried is positive, depends quite crucially on the meaning attached to the Bondi four-vector has led to a long series of analyses concerning the sign of its "time component." Note, however, that a simultaneous validity of the interpretations associated with the two quantities is a nontrivial requirement: The two interpretations are incompatible unless the ADM four-vector equals, in a suitable sense, the sum of the four-momentum radiated away in the form of gravitational waves until any given retarded instant of time, and the Bondi four-vector evaluated at that retarded instant. Although the question of the relation between the two four-vectors was raised[4] immediately after their introduction, very little progress has been made on this issue over the years. The principal difficulty has been the following: Whereas the ADM four-momentum refers to the asymptotic properties of the gravitational field at large spacelike separations from sources, the Bondi four-momentum refers to properties at large null separations.

Recently, however, a new framework was introduced[5,6] to obtain a unified treatment of the properties of the gravitational field in the two asymptotic regimes. In this Letter, we shall use this framework to resolve the issue of the relation between the two four-vectors.

We begin with the notion of asymptotic flatness to be used throughout this analysis.

Definition[7].—A space-time $(\hat{M},\hat{g}_{ab})$ will be said to be asymptotically empty and flat at null and spatial infinity if there exists a space-time, $(\hat{M}, g_{ab})$, with g_{ab} everywhere C^∞ except at a point i^0 where it is $C^{>0}$, together with an imbedding of $\hat{M}$ into M (with which we identify $\hat{M}$ with its image in M) satisfying the following conditions:

(i) $M-\hat{M}=\bar{\mathcal{J}}(i^0)$;

(ii) there exists a function, Ω, on M such that on $\hat{M}$, $g_{ab}=\Omega^2\hat{g}_{ab}$; on $\dot{\mathcal{J}}(i^0)$, $\Omega=0$; on $\dot{\mathcal{J}}(i^0)-i^0$, $\nabla_a\Omega\neq 0$; and, at i^0, $\nabla_a\Omega=0$, $\nabla_a\nabla_b\Omega-2g_{ab}=0$; and

(iii) there exists a neighborhood N of $\dot{\mathcal{J}}(i^0)$ in M, such that (N,g_{ab}) is strongly causal and time orientable; and, in $N\cap\hat{M}$, $\hat{g}_{ab}$ satisfies the vacuum equation $\hat{R}_{ab}=0$.

Denote $\dot{\mathcal{J}}(i^0)-i^0$ by $\mathcal{I}$. It can be shown[6] that $\mathcal{I}$ so defined is endowed with the familiar[2,3] structure of null infinity and that the associated Lie algebra of infinitesimal asymptotic symmetries is the Bondi-Metzner-Sachs (BMS) Lie algebra. Fix any cross section S of $\mathcal{I}^+$. The Bondi four-momentum P_α evaluated at S belongs to the dual of the four-dimensional vector space τ of BMS translations. (Throughout, Greek indices will refer to τ and Latin ones to M.) The expressions available in the literature[2] for P_α hold only in Bondi (conformal) frames. Since these frames are incompatible with conditions imposed at i^0, we first extend these expressions to arbitrary frames:

$$P_\alpha K^\alpha=(1/32\pi)\int_S[{}^*K_{abcd}X_K{}^a 1^b-(q_a{}^m q_b{}^n-\tfrac{1}{2}g_{ab}q^{mn})(\nabla^a l^b)(\alpha R_{mn}-2\Phi\alpha\nabla_m l_n+2\partial_m\partial_n\alpha)\epsilon_{cd}]dS^{cd} \tag{1}$$

for all elements K^α of τ. Here, $X_K{}^a = \alpha n^a \equiv \alpha \nabla^a \Omega$ is the vector field on $\mathscr{I}^+$ representing the BMS translation K^α; $_{ab}$ is the metric induced by g_{ab} on S; ∂ is the derivative operator, and $\epsilon_{cd} = \epsilon_{abcd} n^a l^b$, the alternating tensor on (S, q_{ab}); l^a is the null vector field defined at each point of S by $q_{ab} l^a = 0$ and $l^a n_a = -l$; $\Phi = \frac{1}{4}\nabla_a n^a$ is the divergence of $\mathscr{I}^+$; ϵ_{abcd}, ∇, R_{ab}, and $\Omega\, {}^*K_{abcd} \equiv \epsilon_{abmn} C^{mn}{}_{cd}$ are, respectively, the alternating tensor, the derivative operator, the Ricci tensor, and the dual of the Weyl tensor of g_{ab}. Equation (1) holds in any conformal frame compatible with conditions at $\mathscr{I}^+$. In a Bondi frame, simplifications occur: One has $\Phi = 0$ and $\partial_a \partial_b \propto q_{ab}$, and hence the right-hand side of Eq. (1) reduces to the familiar[2,3] expression involving $\Psi_2{}^0$ and σ^0.

The ADM four-momentum P_a, on the other hand, is a covector in the tangent space T_{i^0}, of i^0.[5,6] Hence, to obtain a relation between P_α and $\tilde{P}_a$, we must set up an isomorphism between τ and τ_{i0}. Fortunately, a *natural* isomorphism does exist![8] To see this, note first that since $\mathscr{I} = \dot{J}(i^0) - i^0$, there is an obvious isomorphism ψ from the action of the BMS group on the space of generators of $\mathscr{I}$ and the action of the Lorentz group in the tangent space of i^0. Since each BMS time translation singles out a SO(3) subgroup of the BMS action on the space of generators and since each time-like direction at i^0 is characterized by a SO(3) subgroup of the Lorentz action on T_{i^0}, the mapping ψ defines an identification between time-like directions in τ and those in T_{i^0}. Finally, since each of τ and T_{i^0} is equipped with a natural Lorentz metric, this identification leads to a unique (metric preserving) isomorphism $\hat{\psi}$ between τ and T_{i^0}.[8] Using this $\hat{\psi}$ we can now ask the desired question: Does $\hat{\psi}[P_\alpha(S) + (\Delta P_\alpha)_{i^0}{}^S]$, with $(\Delta P_\alpha)_{i^0}{}^S$ the four-momentum radiated away between i^0 and S, equal $\tilde{P}_a$?

We can now answer this question. Fix a cross section S_0 of $\mathscr{I}^+$ and a BMS *time* translation K^α. The key idea is to introduce a two-form, ω_{ab}, in the physical space-time with the property that its integral on a two-sphere tends to $P_\alpha K^\alpha$ as the two-sphere converges to S_0, and to $\tilde{P}_a \tilde{K}^a$ as the two-sphere converges to i^0, where $\tilde{K}^a = \hat{\psi}(K^\alpha)$.

It is convenient to work with a conformal frame which satisfies, in addition to the definition, the following conditions: (i) $\Phi \equiv \frac{1}{4}\nabla_a n^a = \text{const}$ on the part of $\dot{J}(i^0)$ enclosed between i^0 and S_0; and, (ii) $\partial_a \alpha \equiv \partial_a(-X_K \cdot l) = 0$ on S_0. (Such frames always exist.) It is then natural to set

$$32\pi\omega_{cd} = {}^*K_{abcd} X_K{}^a l^b - \alpha(q_a{}^m q_b{}^n - \tfrac{1}{2} q_{ab} q^{mn})(\nabla^a l^b)(R_{mn} - 2\Phi \nabla_m l_n)\epsilon_{cd} \tag{2}$$

on S_0. To define ω_{cd} in a neighborhood of $\dot{J}(i^0)$, we must extend fields $X_K{}^a$, α, l_a, q_{ab}, and ϵ_{ab} to this neighborhood. We begin with extensions to $\mathscr{I}^+$. Set, on $\mathscr{I}^+$, $n^a \nabla_a l_b = -\Phi l_b$ and $q_{ab} = g_{ab} + 2n_{(a} l_{b)}$. The fields l_a and q_{ab} now satisfy $l \cdot l = 0$, $q_{ab} l^b = 0$, and, $l \cdot n = -1$ everywhere on $\mathscr{I}^+$ and define a slicing of $\mathscr{I}^+$ by a family of (topological but not necessarily metric) two-spheres. Furthermore, since $n^a \nabla_a \alpha = \Phi \alpha$, on each of these spheres, α is constant and tends to zero as one approaches i^0 along $\mathscr{I}^+$. Next, we extend these fields off $\mathscr{I}^+$. Introduce a smooth foliation of a neighborhood of $\mathscr{I}^+$ by spacelike three-surfaces which intersect $\mathscr{I}^+$ in the above family of two-spheres. Extend the scalar field α to this neighborhood by demanding that it be constant on each of these three-surfaces. It then follows that the foliation admits an extenstion to a neighborhood of $\dot{J}(i^0)$ such that the leaf $\alpha = 0$, passing through i^0, is C^1 and orthogonal to $\tilde{K}^a$ at i^0. Next, consider timelike hypercylinders C_κ defined by $\Omega = \kappa$ (a constant) in this neighborhood. The C_κ's intersect each three-surface $\alpha = \text{const}$ in a family of two-spheres. Let q_{ab} denote the intrinsic metric on these two-spheres and ϵ_{ab} the natural alternating tensor. Define l^a in this neighborhood by $l \cdot l = 0$, $q_{ab} l^b = 0$, and $l \cdot n \equiv l^a \nabla_a \Omega = -1$. Finally, we extend $X_K{}^a$. Denote by K^a any extension of the vector $\tilde{K}^a$ at i^0, which is C^0 at i^0 and smooth elsewhere and which satisfies $K \cdot n = 4\alpha$. Set $X_K{}^a = -\frac{1}{2}\Omega K^a + \frac{1}{4}(K \cdot n) n^a$.[9] (The diffeomorphism generated by this $X_K{}^a$ induces a BMS translation corresponding to K^α on $\mathscr{I}$ and the Spi translation[5,6] corresponding to K^a at i^0. Hence it is a natural extension of $\alpha n^a|_{\mathscr{I}}$.) The two-form ω_{ab} is now defined throughout a neighborhood of $\dot{J}(i^0)$.

On the cylinder C_κ we have

$$\Big(\int_{\tilde{S}_\kappa} - \int_{S_\kappa}\Big)\omega_{ab}\, dS^{ab} = \int_{\Delta_\kappa} \nabla_{[a}\,\omega_{bc]}\, dS^{abc}, \tag{3}$$

where Δ_κ is the part of C_κ bounded by $\tilde{S}_\kappa$ and S_κ (see Fig. 1). We now take the limit as κ tends to zero. Since ω_{ab} is continuous in a neighborhood of S_0 and S_0 is compact, Eqs. (1) and (2) yield

$$\lim_{\kappa \to 0} \int_{S_\kappa} \omega_{ab}\, dS^{ab} = \int_{S_0} \omega_{ab}\, dS^{ab} \equiv P_\alpha K^\alpha. \tag{4}$$

Similarly, using Eq. (2), the expression[5,6] of the ADM four-momentum $\tilde{P}_a$ in terms of the Weyl curvature, and the fact that $\alpha = 0$ on any $\tilde{S}_\kappa$, one obtains:

$$\lim_{\kappa\to 0}\int_{\tilde{S}_0}\omega_{ab}dS^{ab}=\lim_{\kappa\to 0}(1/32\pi)\int_{\tilde{S}_\kappa}{}^*K_{abcd}X_\kappa{}^a l^b dS^{cd} = \tilde{P}_a\tilde{K}^a. \quad (5)$$

Finally, Eqs. (3), (4), and (5) imply that $\lim_{\kappa\to 0}\int_{\Delta_\kappa}\nabla_{[a}\omega_{bc]}dS^{abc}$ must exist. Unfortunately, since we have no assurance that the various fields which enter $\nabla_{[a}\omega_{bc]}$ remain bounded as one approaches i^0 along $\mathscr{I}^+$, and since $\mathscr{J}(i^0)-i^0$ (on which $\nabla_{[a}\omega_{bc]}$ *is* smooth) is not compact, we cannot, in general, conclude that this limit must equal $\int_{\Delta_0}\nabla_{[a}\omega_{bc]}dS^{abc}$; this last integral need not even be finite! Hence, for the class of isolated systems considered so far, although the ADM and the Bondi four-momenta exist, their difference need *not be related,* in a simple way, to the four-momentum carried away by gravitational radiation. However, it is easy to show that if the news tensor,

$$N_{ab}\equiv(q_a{}^m q_b{}^n-\tfrac{1}{2}q_{ab}q^{mn})(R_{mn}-2\Phi\nabla_m l_n),$$

is such that $\alpha^{3-\epsilon}\times(N_{ab}N^{ab})$, with ϵ positive, remains bounded as one approaches i^0 along $\mathscr{I}^+$, or, equivalently, if in a Bondi frame, $u^{1+\epsilon}\times|\partial\sigma^0/\partial u|^2$ remains bounded, one has

$$\lim_{\kappa\to 0}\int_{\Delta_\kappa}\nabla_{[a}\omega_{bc]}dS^{abc} = \int_{\Delta_0}\nabla_{[a}\omega_{bc]}dS^{abc} \equiv(1/32\pi)\int_{\Delta_0}\alpha N_{mn}N^{mn}\epsilon_{[ab}l_{c]}dS^{abc}. \quad (6)$$

The integral on the right is precisely the Bondi flux[2,3] between i^0 and S_0. Since K^α is an *arbitrary* BMS time translation, from Eqs. (3)–(6) we have the following: *Given an isolated system whose underlying space-time satisfies the definition, the ADM and the Bondi four-momenta are related via* $\tilde{P}_a=\psi\{P_\alpha(S_0)+(\Delta P_\alpha)_{i^0}{}^{S_0}\}$ *provided the Bondi news satisfies the above condition as one approaches* i^0 *along* $\mathscr{I}^+$.

Remarks.—(i) That an additional condition is required on the falloff of Bondi news is perhaps to be expected: In the definition, only minimal conditions which ensure that the ADM four-momentum is well defined have been intorduced at i^0. In particular, the metric g_{ab} is assumed to be only C^0 as one approaches i^0 along $\mathscr{I}^+$. It is easy to show that if one demands that the metric connection should admit regular direction-dependent limits not only along spacelike directions (the present $C^{>0}$ condition), but also along null ones, the Bondi news automatically satisfies the

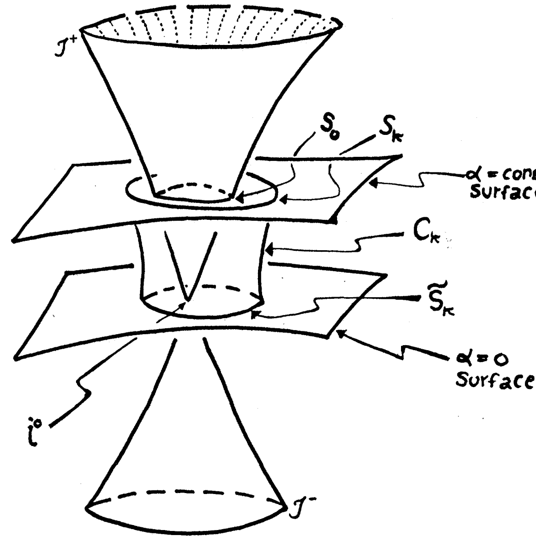

FIG. 1. As κ tends to zero, the cylinders C_κ (defined by $\Omega=\kappa$) converge to $\mathscr{J}(i^0)$, the two-spheres S_κ to the given cross section S_0 of $\mathscr{I}^+$, and the two-spheres $\tilde{S}_\kappa$ to i^0.

required conditions. However, it is not yet clear whether or not this condition on the connection is substantially stronger than the required condition on the news. (ii) The condition on news itself appears to be quite weak. For example, qualitative arguments[10] for systems which coalesce from unbounded distances indicate that such systems will satisfy this condition by a wide margin. More generally, the expression of the total radiated energy suggests that its finiteness will ensure the required falloff of Bondi news in a generic case. (iii) The assumptions in the definition that g_{ab} is C^∞ on $\mathscr{I}$ and that $\hat{g}_{ab}$ satisfies the vacuum equation in $\hat{M}\cap N$ can be substantially weakened. A discussion of this issue as well as of other aspects of the relation between null and spatial infinity will appear elsewhere.

[1]R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 118, 1100 (1960). See also the article in *Gravitation, An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962), and the references contained therein.

[2]H. Bondi, M. G. J. Van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. London, Ser. A 269, 21 (1962).

[3]R. K. Sachs, Proc. Roy. Soc. London, Ser. A 270, 103 (1962); R. Penrose, Phys. Rev. Lett. 10, 66 (1963); J. Winicour, J. Math. Phys. 9, 861 (1968).

[4]See, e.g., R. K. Sachs, in *Relativity, Groups, and*

Topology, edited by C. M. De Witt and B. S. De Witt (Blackie, London, 1964); R. Penrose, Proc. Roy. Soc. London, Ser. A 204, 159 (1965).

[5]A. Ashtekar and R. O. Hansen, J. Math. Phys. 19, 1542 (1978).

[6]A. Ashtekar, in "Einstein Centennial Volume," edited by A. Held *et al.* (Plenum, New York, to be published).

[7]Here, Hawking and Ellis conventions are used for the causal structure of space-time. See, e.g., S. Hawking and G. Ellis, *Large-Scale Strcuture of Space-Time* (Cambridge Univ. Press, London, 1973), Chap. 6. The notion of $C^{>0}$ differentiability as well as the relation of this definition with others is discussed in Ref. 5 or 6.

[8]A. Ashtekar and M. Streubel, "On Angular Momentum of Stationary Gravitating Systems," J. Math. Phys. (to be published).

[9]This extension is suggested by the asympotic behavior of the Killing field in stationary space-times. For details, see A. Ashtekar and A. Magnon-Ashtekar, J. Math. Phys. 20, 793 (1979).

[10]P. D. Sommers, J. Math. Phys. 19, 549 (1978).