[1]J. H. Brewer and K. M. Crowe, Annu. Rev. Nucl. Part. Sci. 28, 239 (1978), and references therein.
[2]J. H. Brewer, D. S. Beder, and D. P. Spencer, Phys. Rev. Lett. 42, 808 (1979).
[3]See L. R. Dalton and L. A. Dalton, Mag. Reson. Rev. 2, 361 (1973), other reviews of double resonance in this journal, and references therein.
[4]T. Coffin, R. L. Garwin, L. M. Lederman, S. Penman, and A. M. Sachs, Phys. Rev. 106, 1108 (1957), and 109, 973 (1958).
[5]D. E. Casperson, T. W. Crane, V. W. Hughes, P. A. Souder, R. D. Stambaugh, P. A. Thompson, H. Orth, G. zu Putlitz, H. F. Kasper, H. W. Reist, and A. B. Denison, Phys. Lett. 59B, 397 (1975); P. A. Thompson, P. Crane, T. Crane, J. J. Amato, V. W. Hughes, G. zu Putlitz, and J. E. Rothberg, Phys. Rev. A 8, 86 (1973); H. G. E. Kobrak, R. A. Swanson, D. Favart, W. Kells, A. Magnon, P. M. McIntyre, J. Roehrig, D. Y. Stowell, and V. L. Telegdi, Phys. Lett. 43B, 526 (1973); D. Favart, P. M. McIntyre, D. Y. Stowell, V. L. Telegdi, R. Devoe, and R. A. Swanson, Phys. Rev. A 8, 1195 (1973), and references in these papers.
[6]J. H. Freed, D. S. Leniart, and J. S. Hyde, J. Chem. Phys. 47, 2762 (1967).
[7]A. Abragam and B. Bleaney, *Electron Paramagnetic Resonance of Transition Ions* (Clarendon, Oxford, 1970).
[8]A more general model in which terms with both frequencies are retained predicts up to seven lines. For the data reported here this more general treatment differs from the one given here by 1% or less for frequency differences and from 7% to 20% for amplitude ratios. A correction for this difference was applied to all data in our quantitative comparisons of experiment and theory. The discussion of the theory which we give here does not refer to this more general model.
[9]B. D. Patterson, A. Hintermann, W. Kündig, P. F. Meier, F. Waldner, H. Graf, E. Recknagel, A. Weidinger, and Th. Wichert, Phys. Rev. Lett. 40, 1347 (1978).
[10]H. Graf, E. Holzschuh, E. Recknagel, A. Weidinger, and Th. Wichert, Hyperfine Interact. 6, 177 (1979).
[11]H. Fischer, Hyperfine Interact. 6, 397 (1979).

Infinite Number of Order Parameters for Spin-Glasses

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This Letter shows that in the mean-field approximation spin-glasses must be described by an infinite number of order parameters in the framework of the replica theory.

From the theoretical point of view, spin-glasses are very important because they describe one of the most simple cases of amorphous material. A good framework to study spin-glasses is the replica theory[1]: One introduces an order parameter $Q_{\alpha,\beta}$ which is the limit, for n going to zero, of an $n\times n$ matrix, zero on the diagonal. In the mean-field approximation the statistical expectation value is different from zero only in the spin-glass phase, at zero magnetic external field. In the Sherrington-Kirkpatrick (S-K) model the mean-field approximation is exact in the thermodynamic limit.[2]

An intriguing feature of this scheme is the necessity of reaching the limit $n=0$ as analytic continuation in n from positive integer n. In the standard treatment of the S-K model one assumes that

$$Q_{\alpha,\beta}=q. \tag{1}$$

Here q is the Edwards-Anderson[1] order parameter and it is understood that Eq. (1) does not hold for $\alpha=\beta$ in which case $Q_{\alpha,\alpha}$ is identically zero.

This *Ansatz* [Eq. (1)] gives results at variance with the computer simulations[3] of the S-K model; moreover, one obtains a negative entropy at zero temperature, i.e., $S(0)=-0.17$ (the entropy of the model must be nonnegative by definition). It has been suggested that the wrong result is due to the fact that the true value of $Q_{\alpha,\beta}$ is not symmetric under permutations of the indices[4]: The parametrization Eq. (1) is not valid and the replica symmetry is broken. Various patterns of symmetry breaking have been proposed[5,6]; in a previous paper[7] it has been noticed that the pattern of symmetry breaking depends on a continuous variable which must be treated as a variational parameter.

If the matrix $Q_{\alpha,\beta}$ is parametrized as a function of three variables, quite good results have been obtained for the S-K model[7]: The agreement with the computer simulations is excellent and the zero-temperature entropy is quite small, with $S(0)=-0.01$. If we generalize this approach, the matrix $Q_{\alpha,\beta}$ becomes a function of many parameters: The three-variable case is only the first step toward this direction.

To clarify this issue it is convenient to study the S-K model, where the matrix $Q_{\alpha,\beta}$ is a function of 1, 3, 5,... parameters, and to study if this sequence of approximations converges. In this note, I have studied the cases with 1, 3, 5,... parameters near the critical temperature T_c where notable simplifications are present. Indeed, near T_c the order parameter $Q_{\alpha,\beta}$ is small and a Taylor expansion in Q is allowed; one finds that the free energy $F(\tau)$ is given by

$$F(\tau)=F(Q_0)[\partial F/\partial Q]_{Q=Q_0}=0$$
$$F(Q)=\lim_{n\to 0} n^{-1}[\tau\,\mathrm{Tr}Q^2-\tfrac{1}{3}\mathrm{Tr}Q^3+\tfrac{1}{4}\sum_{\alpha\beta}Q_{\alpha\beta}{}^4] \qquad (2)$$

with the condition that the Hessian matrix of second derivatives of F has nonnegative eigenvalues. The variable τ is proportional to T_c-T. Other terms proportional to the fourth power of Q could be added without qualitatively changing the results: The term of fourth degree, which we have retained, is the only one which is responsible for the breaking of the replica symmetry.[4-6]

If we use the *Ansatz* that the matrix $Q_{\alpha,\beta}$ belongs to the subspace of matrices for which $n^{-1}\mathrm{Tr}Q^2$ is negative definite [if Q satisfies Eq. (1) $n^{-1}\mathrm{Tr}Q^2=(n-1)q^2$], Eq. (2), restricted to this subspace, becomes

$$F(\tau)=\max[F(Q)]. \qquad (3)$$

The maximation of the free energy with respect to some of the parameters (and not the minimization as usual) is a notable feature of the replica approach[8] and it is connected to the analytic continuation in n up to $n=0$.

The matrix $Q_{\alpha,\beta}$ belongs to a zero-dimensional space, so that it is not evident how to write down the generical matrix of this space. The only known procedure consists in using a simple *Ansatz* for integer n which are analytically continued in n up $n=0$. One possibility is the following:

$$\begin{aligned}
Q_{\alpha,\beta}&=q_0, && I(\alpha/m_1)=I(\beta/m_1);\\
Q_{\alpha,\beta}&=q_1, && I(\alpha/m_1)\neq I(\beta/m_1),\\
& && I(\alpha/m_2)=I(\beta/m_2);\\
Q_{\alpha,\beta}&=q_2, && I(\alpha/m_1)\neq I(\beta/m_1),\\
& && I(\alpha/m_2)\neq I(\beta/m_2);
\end{aligned} \qquad (4)$$

where m_1, m_2, m_2/m_1, and n/m_1 are all integers. $I(x)$ is an integer valued function: Its value is the smallest integer greater or equal to x [e.g., $I(0.4)=1$].

The matrix $Q_{\alpha,\beta}$ depends on five parameters. If $m_2=n$, then Q is independent of q_2 and we recover the case studied in Ref. 7; if $m_1=m_2=n$, then only q_0 is relevant and the replica symmetry is unbroken. It is evident how to generalize Eq. (3) by writing the matrix $Q_{\alpha,\beta}$ as a function of q_i, $i=0,\ldots,N$, and of m_i, $i=1,\ldots,N$ (the total number of parameters being $2N+1$). The free energy $F(Q)$ can be obtained by substituting Eq. (3) in Eq. (2); after some algebra one gets, in the limit $n\to 0$,

$$F(q_i,m_i)=\sum_{i=0}^{N}(m_i-m_{i+1})[-\tau q_i^2-\tfrac{1}{4}q_i^4+\tfrac{1}{3}(2m_i-m_{i+1})q_i^3]+\sum_{i=0}^{N}\sum_{j=1+i}^{N}(m_i-m_{i+1})(m_j-m_{j+1})q_iq_j^2, \qquad (5)$$

where $m_0=1$ and $m_{N+1}=0$, and no restriction is put on the values of the m_i, which are now real numbers. We must look for the stationary point of $F(q_i,m_i)$ as a function of τ; for small τ, one finds, after some lengthy algebra,

$$q_0=\tau+C^{(N)}\tau^2+O(\tau^3),$$
$$q_i=B_i^{(N)}\tau+O(\tau^2),\quad m_i=L_i^{(N)}\tau+O(\tau^2), \qquad (6)$$

where

$$C^{(N)}=\tfrac{3}{2}-(2N+1)^{-2},\quad B_i^{(N)}=[2(N-i)+1]/(2N+1),$$
$$L_i^{(N)}=6i/(2N+1). \qquad (7)$$

If $m_i=m_{i+1}$, $\mathrm{Tr}Q^2/n$ is positive definite. An explicit computation shows that $F(q_i,m_i)$ is a maximum with respect to all the variables, as it should be according to the previous discussion ($L_i^{(N)}>L_{i+1}^{(N)}$).

Using Eqs. (5) and (6) to compute the free energy, we find

$$F(\tau)=\tfrac{1}{3}\tau^3+F_4\tau^4+F_5\tau^5+O(\tau^6), \qquad (8)$$

where F_4 is independent of N ($F_4=\tfrac{1}{4}$) while F_5 depends on N such that

$$F_5^{(N)}=\frac{9}{20}-\frac{1}{5(2N+1)^4}. \qquad (9)$$

If we increase N, the convergence is very fast and the bulk of the corrections are obtained for N as small as 1. The smallness of the corrections going from $N=1$ to higher values of N explains why excellent results have been obtained in Ref. 7, where only the case $N=1$ has been considered. For $N=1$, one finds agreement with the prediction of Thouless, Anderson, and Palmer[9] (TAP) for T greater than 0.2 ($T_c=1$), the computed values of the zero-temperature entropy and the internal energy being, respectively, $S(0) = -0.01$ and $U(0)=-0.7652$. For $N=2$, one finds agreement with the TAP predictions[10] for T greater than 0.1 [$S(0)=-0.003$, $U(0)=-0.7636$]. A simple argument shows that the approximation of keeping N small deteriorates as one goes toward zero temperature.[10]

A check of the consistency of this approach would be the computation of the eigenvalues of the Hessian matrix. According to the suggestion of Ref. 6, at least one eigenvalue should be zero, corresponding to infinite "replicon susceptibility": All other eigenvalues should be nonnegative.

A simple way to code the information contained in the parameters q_i and m_i consists in defining the function $q(x)$ as

$$q(x)=q_i, \quad m_i<x<m_{i+1}. \tag{10}$$

For finite N, $q(x)$ is a piecewice constant function; in the limit $N\to\infty$, we find

$$q(x)=\tfrac{1}{3}x+O(\tau^2), \quad x\leq 3\tau; \qquad q(x)=\tau+O(\tau^2), \quad x\geq 3\tau. \tag{11}$$

The infinite number of order parameters that we have introduced is equivalent to a function on the interval 0–1. Replica symmetry is unbroken only if this function is a constant. The physical interpretation of this function is unclear at the present moment and it should be the subject of further investigations. An elementary computation in the Sherrington-Kirkpatrick model shows that the magnetic susceptibility χ and the internal energy U are given, respectively, by

$$\chi=\int_0^1 B[1-q(x)]dx, \qquad U=\int_0^1\{B[q^2(x)-1]/2\}dx. \tag{12}$$

The correct identification of the Edwards-Anderson order parameter $q_{\rm ph}=\langle\langle S\rangle^2\rangle$ is problematic in this approach: It has been suggested that

$$q_{\rm ph}=\max_x[q(x)].$$

If this identification is correct, the breaking of the replica symmetry implies that the Fischer relation,[11] $\chi=\beta[1-q_{\rm ph}]$, does not hold. In the framework of the TAP[9] approach serious doubts have also been case on the validity of the Fischer relation.[12] Recent Monte Carlo simulations of the S-K model strongly suggest that the Fischer relation is not satisfied.[13]

[1]S. F. Edwards and P. W. Anderson, J. Phys. F 5, 965 (1975).
[2]D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1972 (1975).
[3]D. Sherrington and S. Kirkpatrick, Phys. Rev. B 17, 4385 (1978).
[4]J. R. L. de Almeida and D. J. Thouless, J. Phys. A 11, 983 (1978).
[5]J. Blandin, J. Phys. (Paris), Colloq. 39, C-6, 1568 (1978).
[6]A. J. Bray and M. A. Moore, Phys. Rev. Lett. 41, 1068 (1978).
[7]G. Parisi, to be published.
[8]J. Chalupa, Phys. Rev. B 17, 4335 (1978).
[9]D. J. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. 35, 593 (1977).
[10]G. Parisi, to be published.
[11]K. H. Fischer, Phys. Rev. Lett. 34, 1438 (1975).
[12]A. J. Bray and M. A. Moore, to be published.
[13]G. Parisi, to be published.