Diagonalization of the Chiral-Invariant Gross-Neveu Hamiltonian

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The Hamiltonian of the chiral-invariant Gross-Neveu model is diagonalized, without approximation, using a modified Bethe *Ansatz.*

The chiral-invariant Gross-Neveu model, with formal Lagrangian

$$\mathcal{L}=i\bar{\psi}_a \partial\!\!\!/ \psi_a+g[(\bar{\psi}_a\psi_a)^2-(\bar{\psi}_a\gamma^5\psi_a)^2], \quad a=1,2,\ldots,N_c \tag{1}$$

has a number of interesting features. It is asymptotically free[1] and exhibits dynamical mass generation[1-3] without the presence of a Goldstone boson.[4] The chiral symmetry remains unbroken,[5] and the massive physical particles have zero chirality. This peculiarly two-dimensional phenomenon involves the existence of a massless excitation which decouples from the rest of the spectrum.

In this Letter we seek to diagonalize the Hamiltonian H, the integral of

$$\mathcal{H}=-i(\psi_{a+}{}^*\partial_x\psi_{a+}-\psi_{a-}{}^*\partial_x\psi_{a-})+4g\psi_{a+}{}^*\psi_{b-}{}^*\psi_{b+}\psi_{a-}, \tag{2}$$

where we have chosen γ^5 to be diagonal and have Wick ordered the products of fields. The model is thought to possess an infinite number of conserved charges,[6] so that in any scattering process the sets of incoming and outgoing momenta coincide. It is thus natural to use a Bethe *Ansatz*[7] for possible eigenstates of H, as was done recently in the case of the massive Thirring model.[8]

We assume canonical anticommutation relations for the cut-off, unrenormalized fields, and construct a Fock representation with cyclic state $|0\rangle$ (the drained Dirac sea) defined by

$$\psi_{a+}|0\rangle=0=\psi_{a-}|0\rangle. \tag{3}$$

The physical vacuum will be obtained by "filling the sea." Consider states of the form

$$|F,\xi\rangle=\int d^Nx\sum_{\beta,a}F(x_1,\ldots,x_N;\ \beta_1,\ldots,\beta_N)\xi(a_1,\ldots,a_N)\prod_i^N\psi_{a_i\beta_i}{}^*(x_0)|0\rangle. \tag{4}$$

Here $\beta_i=\pm1$ is the chirality and $\xi(a)$ is the color wave function. In order for $|F,\xi\rangle$ to be an eigenstate of H, $F(x;\beta)$ must be an eigenfunction of the N-particle Hamiltonian

$$h=-i\sum_{i=1}^N\beta_i\partial_i-4g\sum_{i,j}\delta(x_i-x_j)P^{ij}[\tfrac{1}{2}(1-\beta_i\beta_j)], \tag{5}$$

where P^{ij} is an operator which exchanges β_i and β_j. Although h does not involve color, the presence of $\xi(a)$ allows nontrivial permutation symmetry $[m_1,\ldots,m_l]$, $m_i\le N_c$, for $F(x;\beta)$.

To diagonalize h, we employ a method devised by Yang.[9] We first divide configuration space into regions labeled by permutations $Q\propto S_N$. In the interior of region Q, defined by $0\le x_{Q1}\le x_{Q2}\le\cdots\le x_{QN}\le L$, the particles are free and we write F as a superposition of plane waves labeled by N momenta k_i and chiralities α_i:

$$F(x;\beta)=\sum_{P\propto S_N}\xi_P(Q)\exp(i\sum_{j=1}^N k_{Pj}x_{Qj})\prod_{l=1}^N\delta_{\alpha_{Pl}\beta_{Ql}} \tag{6}$$

The corresponding energy and momentum are

$$E=\sum_{i=1}^N\alpha_ik_i,\quad P=\sum_{i=1}^N k_i. \tag{7}$$

Note that $N^{\pm}$, the numbers of $\alpha_i = \pm 1$, are conserved quantum numbers since H is chiral invariant.

In order for F to have proper discontinuities at the boundary between neighboring regions Q and Q' ($Qa = Q'b$, $Qb = Q'a$, $b = a+1$; $Qm = Q'm$, $m \neq a, b$) it is sufficient that the coefficients $\xi_P(Q)$, regarded as ($N!$)-component vectors, satisfy

$$\xi_{P'} = Y_{ij}{}^{ab}\xi_P, \quad i = Pa = P'b, \quad j = Pb = P'a \tag{8}$$

with

$$Y_{ij}{}^{ab} = \begin{cases} \dfrac{1-g^2}{1+g^2}\mathcal{P}^{ab} + \dfrac{ig}{1+g^2}(\alpha_i - \alpha_j)\mathcal{I}, & \alpha_i \neq \alpha_j \\ \mathcal{I}, & \alpha_i = \alpha_j. \end{cases} \tag{9}$$

Here $\mathcal{P}^{ab}$ interchanges $\xi_p(Q)$ and $\xi_p(Q')$ and $\mathcal{I}$ is the identity operator.

Imposing periodic boundary conditions leads to an eigenvalue equation for $\Phi = \mathrm{sgn}(Q)\xi_{\mathrm{identity}}(Q)$:

$$X_{j+1,j}{}' \cdots X_{Nj}{}' X_{1j}{}' \cdots X_{j-1,j}\Phi = \exp(ik_j L)\Phi, \tag{10}$$

where

$$X_{ij}{}' = e_{ij}\,\frac{i(\alpha_i - \alpha_j)\mathcal{I} + c\mathcal{P}^{ij}}{i(\alpha_i - \alpha_j) + c},$$
$$e_{ij} = \begin{cases} \exp(i\alpha_j\varphi), & \alpha_i \neq \alpha_j \\ \exp(i\alpha_j\pi), & \alpha_i = \alpha_j, \end{cases} \tag{11}$$
$$c = 4g/(1-g^2) = 2\tan\varphi.$$

We shall find that c is the effective coupling constant.

We now restrict ourselves to $N_c = 2$. To solve (10), we consider $\mathcal{P}^{ij}$ acting on N spins forming a cyclic chain. If $1 \leq y_1 < \cdots < y_M \leq N$ are the coordinates of M down spins, we find a solution of the modified Bethe form[9]

$$\Phi(y_1, \ldots, y_M) = \sum_{P \in S_M} A_P f(\Lambda_{P1}, y_1) \cdots f(\Lambda_{PM}, y_M), \tag{12}$$

with

$$f(\Lambda, y) = \prod_{j=1}^{y-1} \frac{i(\alpha_j - \Lambda) + c'}{i(\alpha_{j+1} - \Lambda) - c'}, \quad c' = \tfrac{1}{2}c, \tag{13}$$

$$e^{ik_jL} = \prod_{i \neq j}^{N} e_{ij} \prod_{\gamma=1}^{M} \frac{i(\alpha_j - \Lambda_\gamma) + c'}{i(\alpha_j - \Lambda_\gamma) - c'}, \tag{14}$$

provided that the M numbers Λ_γ are distinct and satisfy

$$N^+\theta(2\Lambda_\gamma - 2) + N^-\theta(2\Lambda_\gamma + 2) = -2\pi J(\Lambda_\gamma) + \sum_{\delta=1}^{M}\theta(\Lambda_\gamma - \Lambda_\delta), \tag{15}$$

where

$$\theta(x) = -2\tan^{-1}(x/c), \quad -\pi \leq \theta < \pi$$

and the $J(\Lambda_\gamma)$ are half-integers (integers) when M is even (odd). From (14), the allowed momenta of the bare particles are

$$k_j = 2\pi L^{-1}n_j + \alpha_j N^{(-\alpha_j)}L^{-1}\varphi + \alpha_j(N^{(\alpha_j)} - 1)L^{-1}\pi + L^{-1}\sum_{\gamma=1}^{M}[\theta(2\Lambda_\gamma - 2\alpha_j) - \alpha_j\pi], \tag{16}$$

so that the total energy is

$$E = L^{-1}[2\pi\textstyle\sum_j \alpha_j n_j + 2N^+N^-\varphi + N^+(N^+ - 1)\pi + N^-(N^- - 1)\pi + N^+\sum_\Lambda\theta(2\Lambda - 2) - N^-\sum_\Lambda\theta(2\Lambda + 2) - NM\pi]. \tag{17}$$

To render the sums over j finite, we cut off $2\pi L^{-1}|n_j|$ at K. We now proceed to find the ground state and various excited states of the Hamiltonian.

Vacuum state.—The state of lowest energy is a color singlet of vanishing chirality, with $N^{\pm}$ and M all equal to $\frac{1}{2}N$. The Λ_γ are all real, and the $J(\Lambda_\gamma)$ in (15) are consecutive half-integers for $N^{\pm}$ assumed to be even. The n_j quantum levels are filled in such a way as to minimize the energy. From (15) with $N^{\pm} = \frac{1}{2}N$, it follows that in the continuum limit the density σ_0 of Λ_γ on the real line must satisfy

$$2\pi\sigma_0(\Lambda) = -2c\int d\Lambda'\,\sigma_0(\Lambda')[c^2 + (\Lambda - \Lambda')^2]^{-1} + 2cN[c^2 + 4(\Lambda - 1)^2]^{-1} + 2cN[c^2 + 4(\Lambda + 1)^2]^{-1}. \tag{18}$$

Introducing Fourier transforms in (18), one obtains

$$\sigma_0(p) = N\cos p\,(2\cosh c'p)^{-1}. \tag{19}$$

One can now calculate the vacuum energy from (17):

$$E_0 = -(K + \pi/L)N + \tfrac{1}{4}N^2L^{-1}(\pi + 2\hat{\varphi}), \tag{20}$$

where

$$\hat{\varphi} = \varphi + N^{-1}\int d\Lambda\, \sigma_0(\Lambda)[\theta(2\Lambda - 2) - \theta(2\Lambda + 2)] - \tfrac{1}{2}\pi.$$

Minimizing the energy then yields, for the number of bare particles (dropping terms of order L^0),

$$N = 2KL(\pi + 2\hat{\varphi})^{-1}. \tag{21}$$

Massless excitations.—A massless, colorless excitation with vanishing chirality may be obtained by simply raising one bare particle in the Dirac sea to an empty level above it. This excitation is clearly the analog of the scalar boson in the massless Thirring model.

Massive excitation: color spinor.—We now consider a state with unit chirality and color spin $\frac{1}{2}$, increasing N^+ by one while leaving N^- and M at their vacuum values. There are now $M+1$ allowed values for $J(\Lambda_\gamma)$, but only M Λ_γ. Thus there is inevitably a "hole" in the sequence of $J(\Lambda_\gamma)$ values, say at Λ^0. From (15), the Fourier transform of the density $\sigma(\Lambda)$ for this state is found to be

$$\sigma(p) = \sigma_0(p) + \exp(-ip)(2\cosh c'p)^{-1} - \exp(-ip\Lambda^0)[1 + \exp(-c|p|)]^{-1}. \tag{22}$$

The energy and momentum, relative to the vacuum, are readily computed:

$$\begin{aligned} E' &\equiv E - E_0 = |q| + NL^{-1}\tan^{-1}[\cosh(\pi\Lambda^0/c)/\sinh(\pi/c)], \\ P' &\equiv P - P_F = q + NL^{-1}\tan^{-1}[\sinh(\pi\Lambda^0/c)/\cosh(\pi/c)], \end{aligned} \tag{23}$$

where $q = 2\pi L^{-1} n_0 + K$, with n_0 the additional level, and $P_F = -2\hat{\varphi}K(\pi + 2\hat{\varphi})^{-1}$ is the effective Fermi momentum.

We see that the energy and momentum have two contributions: a massless chiral excitation analogous to that in the Thirring model,[10] and a massive particle with color spin $\frac{1}{2}$ and chirality zero. Its mass is

$$m = NL^{-1}\tan^{-1}\{1/[\sinh(\pi/c)]\} \to 4K\pi^{-1}\exp(-\pi/c) \quad \text{for } K \to \infty,\ c \to 0. \tag{24}$$

With both m and the rapidity $\chi = \pi\Lambda^0/c$ taken to be cutoff independent, we obtain the relativistic spectrum

$$\begin{aligned} E' &= |q| + m\cosh\chi, \\ P' &= q + m\sinh\chi. \end{aligned} \tag{25}$$

Formula (24) exhibits both the dimensional transmutation and asymptotic freedom characteristic of the model. As $K \to \infty$, the dimensionless coupling constant vanishes logarithmically, and the resulting theory is parametrized only by m.

Massive excitations: color triplet and singlet.—A color triplet state with $N^\pm = \frac{1}{2}N = M+1$, as well as a color singlet state with $N^\pm = \frac{1}{2}N = M$, are obtained by putting two holes (at Λ^1 and Λ^2) in the sequence of $J(\Lambda_\gamma)$ values. The singlet case requires in addition a conjugate pair (two-string)[11] of complex Λ_γ with $\mathrm{Im}\Lambda_\gamma = \pm ic'$. The energy and momentum may be calculated as before (for $K \to \infty$):

$$\begin{aligned} E' &= m\cosh\chi_1 + m\cosh\chi_2, \\ P' &= m\sinh\chi_1 + m\sinh\chi_2. \end{aligned} \tag{26}$$

We have succeeded in diagonalizing the Hamiltonian of the cut-off chiral-invariant Gross-Neveu model, and have obtained the spectrum in the limit of infinite cutoff. The vacuum and low-lying excited states have been described above; other energy eigenstates, with higher chirality, color spin, or numbers of massive and massless particles may be obtained similarly. We have concentrated on the case $N_c = 2$, but our method can be extended to arbitrary N_c using Sutherland's generalization[12] of the Bethe-Yang *Ansatz*. This will be presented elsewhere.

Note added: L. D. Fadeev has informed us that A. Belavin has treated a related chiral-invariant model obtaining similar results.

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Experimental Characteristics of Dynamical Pseudo Goldstone Bosons

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The hypothetical existence of new color interactions, which participate in the spontaneous breaking of the weak-interaction group, will in general lead to relatively light composite pseudo Goldstone bosons. Their production and decay characteristics are analyzed to be close to, yet actually distinguishable from, those of the elementary Higgs bosons of the Weinberg-Salam model.

The usual implementation of the Goldstone-Higgs mechanism of spontaneous symmetry breaking, via elementary spin-0 fields, is one of the less attractive features of conventional quantum flavordynamics (QFD). And the situation is acute in attempts to unify QFD and quantum chromodynamics (QCD) into a single gauge theory. In such grand unified theories, one is obliged to introduce a multitude of Higgs fields with judiciously contrived couplings; the essential simplicity of the gauge-theoretic approach is, thereby, irretrievably lost. It has been suggested,[1-4] therefore, that one discard elementary Higgs fields altogether, and seek a dynamical mechanism for symmetry breakdown.

In the simplest dynamical mechanism, the requisite Goldstone bosons, which furnish the longitudinal degrees of freedom for massive gauge fields, are bound states of a new species of quark, whose superstrong gauge interactions (generated by gauging a color′ degree of freedom and described by a theory hereinafter called QC′D) spontaneously break chiral symmetry.

The color′ quarks are likely to come in several flavors, in which case there will be several light pseudo Goldstone bosons as well. In this Letter we observe that these particles may be as light as 10 GeV, that they will be relatively pointlike (of size 1 TeV^{-1}), and will have production and decay modes that are determined by partial conservation of axial-vector-current arguments and hence are fairly model independent. Their signatures are, crudely speaking, similar to those of elementary Higgs particles; consequently, we stress the differences. If spin-0 weakly interacting particles are discovered with masses of tens of GeV, the question of composite versus elementary need not wait until energies of 1 TeV probe the possible bound-state structure.

The color′ degree of freedom, first introduced by Weinberg,[3] may be dubbed "hypercolor."[4] The terminology is convenient, with words such as hyperquark, hyperpion, and hyper-σ having an obvious meaning. We take the weak and electromagnetic interactions of the hyperquarks to be isomorphic to those of ordinary quarks so that a flavor doublet such as (u', d') transforms as (u, d) under the electroweak group.

The existence of hyperquarks is, of course, logically independent of the existence of elementary Higgs particles; we do not rule out the possibility that there may exist both hyperpions and

