$K_2 = |\vec{E}(\vec{X})|^2[\omega^2 - \Omega^2(\vec{X})]^{-1}$. We must not be too close to the resonance because there the perturbation expansion breaks down. If in addition we assume $\omega \ll \Omega(\vec{X})$ we obtain the Alfvên confinement.[10] (ii) In the electrostatic limit $\vec{E}(\vec{X}) = -i\vec{k}(\vec{X})\varphi(\vec{X})$,

$$K_2 + |\varphi(\vec{X})|^2 \sum_{l=-\infty}^{+\infty} \left\{ \frac{k_\parallel^2(\vec{X}) J_l^2}{[\omega - l\Omega(\vec{X}) - k_\parallel(\vec{X})P]^2} + \frac{l \partial J_l^2/\partial\mu}{\omega - l\Omega(\vec{X}) - k_\parallel(\vec{X})P} \right\}. \tag{8}$$

In forthcoming communications we plan to apply this formalism to study plasma kinetic theory and nonlinear wave interaction.

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Reconsideration of Quasilinear Theory

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It is shown that even within the quasilinear framework, mode-coupling terms give a zero-order contribution to the growth rate at least for one-dimensional Langmuir waves.

In this Letter, we reconsider the quasilinear theory[1] and its validity for describing the resonant interaction of waves with particles. For simplicity, we limit ourselves to the case of 1D Langmuir turbulence in which ions are treated as an immobile neutralizing background. Quasilinear theory lies upon the assumption that the correlation time τ_c of the electric field seen by a resonant particle is small as compared with the evolution time of averaged quantities. It is generally admitted that the previous assumption allows one to neglect the mode-mode coupling terms, leading them to the use of the linear dispersion equation for the evolution of the electric field. Renormalized theories[2] take into account some classes of nonlinear terms but in the limit $\tau_c \to 0$, they do not lead to significant corrections. In the first part of this Letter, we compute the field spectrum from the two-point, double-time correlation function of the particles. The growth rate is found to be modified with respect to the usual quasilinear result at the lowest order. The modification is a consequence of wave emission by strongly correlated resonant particles. In order to understand why this effect is not obtained in the classical quasilinear theory, we show in the second part of the Letter that mode-mode coupling terms are actually not negligible when the resonant particles are taken into account in the computation of the mode-coupling coefficients.

We have obtained an equation which gives the correlation function of the particles in terms of the field correlation function by using diagrammatic methods which will be published elsewhere. Here we present a heuristic derivation which makes use of conventional techniques of plasma turbulence theory.

The field correlation function can be written

$$\langle E(x,t)E(x',t')\rangle = \int dk\, N_k(t,t')\exp\{i[k(x-x')-\omega_k(t-t')]\}$$

and we assume that N_k is a slowly varying function of t and t'. We define the particle correlation function g by

$$g(x-x',v,v',t') = \langle f(x,v,t)f(x',v',t')\rangle - \langle f(x,v,t)\rangle\langle f(x',v',t')\rangle,$$

where the angular brackets denote an ensemble average. By integrating the Vlasov equation for the fluctuating part of the distribution along the exact trajectories, we obtain

$$g(x-x',v,v',t,t') = (q/m)^2\int_0^t du\int_0^{t'}\int dk\, N_k(u,u')\exp[-i\omega_k(u-u')] \times\langle\exp\{ik[x(u)-x(u')]\}\rangle\langle\partial f(u)/\partial v\rangle\langle\partial f(u')/\partial v'\rangle, \quad (1)$$

where $x(u)$ and $x(u')$ are the solutions of the motion equations at time u and u' such that $x(u=t)=x$, $v(u=t)=v$ and $x(u'=t')=x'$, $v(u'=t')=v'$. For obtaining Eq. (1), we have replaced the exact propagator for the two particles by the averaged one, as it is usually done.[2] This result is a double-time generalization of the expression that Dupree obtained[3] for the single-time, two-point correlation function.

In order to evaluate the first angularly bracketted term in the right-hand side of Eq. (1), we notice that a particle with a given velocity v is reasonantly coupled with modes having phase velocities ω_k/k such that $|\omega_k/k - v| \lesssim (D/k)^{1/3}$, where D is the usual quasilinear diffusion coefficient. Consequently, we consider two cases. Firstly, if $|v-v'| \gg (D/k)^{1/3}$, the orbits are uncorrelated and in Eq. (1) we are allowed to set

$$\langle\exp\{ik[x(u)-x(u')]\}\rangle = \langle\exp[ikx(u)]\rangle\langle\exp[-ikx(u')]\rangle. \quad (2)$$

In this case, provided that τ_c is small, we can write

$$g(k,v,v',t,t') = (q/m)^2\exp[-i\omega_k(t-t')]R_k(v,t)R_{-k}(v',t')N_k\langle\partial f/\partial v\rangle\langle\partial f/\partial v'\rangle, \quad (3)$$

where $g(k)$ is the Fourier transform of g with respect to $x-x'$. The operator R_k is defined by

$$R_k(v,t)\varphi(v,t) = \int_0^t du\exp[i(kv-\omega_k)u - (k^2D)u^3/3]\varphi(v,t-u).$$

This result could also have been obtained by use of the resonance-broadening propagator to compute the perturbation of the distribution function $\tilde{f}(x,v,t)$ induced by the electric field, and by making an ensemble average of the product $\tilde{f}(x,v,t)\tilde{f}(x',v',t')$.

We can already use these simple results to determine the mode frequencies. From Poisson equation, we find that

$$k^2N_k(t,t')\exp[-i\omega_k(t-t')] = (q^2/\epsilon_0^2)\int g(k,v,v',t,t')dv\,dv'. \quad (4)$$

We now take into account the weak turbulence ordering by assuming that the velocity gradient of the distribution function is small for resonant particles. More precisely, we set $f=f_p+f_h$, where f_p is the distribution of cold plasma and f_h is the distribution function of hot particles which can interact resonantly with the waves. The linear growth rate γ_k^L depends on f_h and we assume that the inequality

$$\eta = \gamma_k^L/\omega_{pe} \ll 1 \quad (5)$$

is satisfied.

We insert the result of Eq. (3) into Eq. (4) and look for solutions such that

$$N_k(t,t') = N_k\exp[-i\delta\omega_k(t-t') + \gamma_k(t+t')].$$

If ω_k is taken to be the solution of the linear dispersion equation

$$\epsilon(k,\omega_k)=1-\frac{q^2}{m\epsilon_0}\int \mathrm{P}\frac{1}{kv-\omega_k}\left(\frac{\partial\langle f\rangle}{\partial v}\right)dv,$$

where P denotes the principal values, Eq. (4) shows that $\delta\omega_k$ is of order η^2. However, we find that γ_k cannot be obtained from Eq. (4), because it cancels out at the lowest order in η. In order to compute γ_k, we use once more the Poisson equation to write

$$\left\langle\left[ikE_k(t)-\frac{q}{\epsilon_0}\int_p f_k(t)dv\right]\left[ikE_{-k}(t')+\frac{q}{\epsilon_0}\int_p f_{-k}(t')dv'\right]\right\rangle=-\frac{q^2}{\epsilon_0^{\,2}}\int_h g(k,v,v',t,t')dv\,dv', \tag{6}$$

where in the right-hand side the integration is restricted to hot particles. We assume that the cold plasma behaves linearly so that the left-hand side contains the product of the two dielectric responses of the cold plasma at t and t'. As we shall see, $g(k)$ will appear as a sum of terms oscillating like $\exp[-ip\omega_k(t-t')]$. Only the components $p=\pm 1$ will be resonant and thus will lead to a significant contribution to the time variation of the wave spectrum N_k. We divide the integration range of the hot particles into the uncorrelated domain (u), where $|v-v'|\gg(D/k)^{1/3}$, and the correlated domain (c), where $|v-v'|\lesssim(D/k)^{1/3}$. Using Eq. (3) in the domain u and the previous definition of ω_k, we obtain, up to order η^3,

$$k^2\left|\frac{\partial\epsilon}{\partial\omega_k}\right|^2\left[\frac{\partial^2}{\partial t\,\partial t'}N_k(t,t')\right]_{t=t'}=k^2\left|\frac{\partial\epsilon}{\partial\omega_k}\right|^2\gamma_k^{\,2}N_k\exp 2\gamma_k t=\frac{q^2}{\epsilon_0^{\,2}}\int_{(c)}g_1(k,v,v',t,t)dv\,dv', \tag{7}$$

where $g_1(k,v,v',t,t')$ denotes the $p=1$ component of $g(k,v,v',t,t')$. We now have to compute $g(k,v,v',t,t)$ for correlated particles, which corresponds to the second case, $|v-v'|\lesssim(D/k)^{1/3}$. As pointed out by Dupree,[3] the result given by Eq. (3) is no longer valid since, when $|v-v'|\lesssim(D/k)^{1/3}$, the two particles interact resonantly with the same Fourier component of the electric field. We need only the single-time correlation function which is provided by the Fokker-Planck equation for $\langle f(x,v,t)f(x',v',t)\rangle$. We obtain

$$[\partial_t+v_-\partial_{x_-}-2D(v_+,t)(1-\cos k_+x_-)(\partial_{v_-2})^2]g(x_-,v_-,v_+,t,t)=2D(v_+,t)(\cos k_+x_-)(\partial\langle f\rangle/\partial v_+)^2, \tag{8}$$

where $v_-=v-v'$, $x_-=x-x'$, $v_+=(v+v')/2$, and k_+ is the positive root of the equation $k_+v_+=\omega_{k_+}$. In Eq. (8), we have neglected the v_+ dependence of the diffusion coefficient $D(v_+,t)$ as compared with the v_- dependence of g because the solution will exhibit a fast variation in v_- close to $v_-=0$. Equation (8) is the Dupree equation for the correlation function, namely Eq. (49) of Ref. 3, in which we have made a weak-turbulence assumption by assigning a single mode frequency to a given k.

When the linear growth rate $\gamma_k^{\,L}$ is larger than $(k^2D)^{1/3}$, we can neglect the diffusion term in the left-hand side of Eq. (8), and Eq. (8) provides the linear result for g. Inserting then this solution into Eq. (7), we recover the usual linear growth rate, namely $\gamma_k^{\,2}=(\gamma_k^{\,L})^2$, where $\gamma_k^{\,L}$ is the Landau growth rate.

On the other hand, for $\gamma_k^{\,L}\ll(k^2D)^{1/3}$, we can neglect the time derivative of g in Eq. (8). It is then possible to solve exactly Eq. (8) and write the solution as follows:

$$g=\left[\sum_{p=-\infty}^{+\infty}g_p(v_-)\delta(k-pk_+)\right]2D(v_+,t)(\partial\langle f\rangle/\partial v_+)^2, \tag{9}$$

with

$$g_p(v_-)=(2k^2D)^{-1/3}\Big\{\sum_{n=1}^{\infty}[J_{n-p}(n)+J_{n+p}(n)][J_n(n)/n]\int_0^\infty\cos\mu w\exp(-\mu^3/3n)d\mu$$
$$+i\sum_{n=1}^{\infty}[J_{n-p}(n)-J_{n+p}(n)][J_n(n)/n]\int_0^\infty\sin\mu w\exp(-\mu^3/3n)d\mu\Big\},$$

where $w=(2D/k)^{-1/3}v_-$ and $J_n(x)$ is the Bessel function of order n. Like in the resonance-broadening theory, the correlation function is peaked around $v_-=0$ with a characteristic width $\Delta v_t=(D/k)^{1/3}$. As shown in Eq. (12) of Ref. 3, for $t\neq t'$, the terms $g_p(v_-)$ oscillate like $\exp[-ipkv_+(t-t')]=\exp[ip\omega_k(t-t')]$. Thus, as explained before, we are interested only in g_1 and g_{-1} which can be approximated by

$$g_1=g_{-1}=\tfrac{1}{2}\pi A\,\delta(k_+v_-), \tag{10}$$

where $A = 4\sum_{n=1}^{\infty}[J_n^2(n)/n] = 1.668$ instead of $A = 1$ in the linear or resonance-broadening theory. This modification results from the x_- dependence of the diffusion term in the left-hand side of Eq. (8). It generates harmonics of k_+ which modify the fundamental terms by recoupling. It strongly recalls O'Neil's arguments[4] for keeping nonlinear terms when computing the damping of monochromatic waves and reflects the importance of partial particle trapping of resonant particles.[5]

Using Eq. (10) to compute the right-hand side of Eq. (7), we obtain

$$\gamma_k^{\,2} = A(\gamma_k^{\,L})^2. \qquad (11)$$

Thus, when $\gamma_k^{\,L} \ll (k^2D)^{1/3}$, the time evolution of the field spectrum is no longer given by the linear theory. In this regime, the nonlinear motion of resonant particles leads to a finite modification of the growth rate because it takes place on a time scale of the order of $(k^2D)^{-1/3} \ll \gamma_k^{\,-1}$ and then cannot be ignored. A rigorous demonstration based upon diagrammatic methods limits the validity of these conclusions to weakly dispersive waves such that

$$(d^2\omega_k/dk^2)[k^2D)^{1/3}/v(k)]^2 \ll \gamma_k \ll (k^2D)^{1/3}, \qquad (12)$$

where $v(k) = \omega_k/k$ is the velocity which is resonant with wave k.

These results contradict the usual quasilinear theory. Energy conservation is satisfied if the equation for $\langle f\rangle$ is modified.[3] Similar conclusions have been obtained by Bakai and Sigov.[5]

We indicate now why the usual methods for establishing quasilinear theory are questionable. In all derivations of quasilinear theory, the contribution of resonant particles is deleted in the computation of the matrix elements of the mode-mode coupling terms.[6] Within this approximation, these terms are shown to be negligible in the limit $\tau_c \to 0$. Let us show that as a result of resonant particles, the mode-mode coupling terms give actually a contribution of the same order as the linear term. For instance, in the evolution of the wave intensity N_k, we compare the linear term $2\gamma_k^{\,L}N_k$ with the four-waves coupling term

$$C_4 = \int dk_1\,dk_2\,M_{k_1,k_2,k-k_1-k_2}N_{k_1}N_{k_2}N_{k-k_1-k_2}.$$

The matrix M is the sum of several matrices related to different elementary processes. We find that one of them leads to secularities when the resonant particles are taken into account. The corresponding matrix $M^{\rm sec}$ may be written as

$$M^{\rm sec}_{k_1,k_2,k-k_1-k_2} = \frac{-2|\gamma_k|}{[(\sum\omega)^2+\gamma_k^{\,2}]}\,\frac{(q/\epsilon_0)^2(q/m)^6}{k^2\,|\partial\epsilon/\partial\omega_k|^2}\,|\varphi_{k_1,k_2,k-k_1-k_2}|^2 \qquad (13)$$

with $\sum\omega_{k_1}+\omega_{k_2}+\omega_{k-k_1-k_2}-\omega_k$; the quantity φ_{k_1,k_2,k_3} is defined by

$$\varphi_{k_1,k_2,k_3} = \int\frac{dv}{[\omega_{k_1}+\omega_{k_2}+\omega_{k_3}-(k_1+k_2+k_3)v+i\alpha]}\,\partial_v\left[\frac{1}{\omega_{k_2}+\omega_{k_3}-(k_2+k_3)v+i\alpha}\,\partial_v\left(\frac{1}{\omega_{k_3}-k_3v+i\alpha}\,\partial_v\langle f\rangle\right)\right]. \qquad (14)$$

In Eq. (14), $\alpha > 0$ provides a prescription of contour for v integration. The resonant particles contribute to the quantity $\varphi_{k_1,k_2,k-k_1-k_2}$ by terms of the form

$$\frac{|k|(\partial\langle f\rangle/\partial v)_{v(k)}}{[-(k-k_1)v(k)+i\alpha][-(2k-k_1-k_2)v(k)+i\alpha]^3}.$$

We notice that the product $\varphi\varphi^*$ yields, in the expression of $M^{\rm sec}$, multiple poles in $k-k_1$ and $2k-k_1-k_2$ which are on both sides of the real axis. Consequently the k_1 and k_2 integrals will be divergent as $\alpha \to 0$, indicating a secular behavior of this mode-mode coupling term. Moreover, it can be checked that this term does not belong to the mode-mode coupling terms which are summed up when using renormalized propagator. The secular behavior is removed by keeping the growth rates in the resonant denominators and with use of a renormalized propagator which includes resonance broadening. In order to estimate the magnitude of this term, it is sufficient to replace α by $\max[\gamma_k, (k^2D)^{1/3}]$. We then obtain, for the order of magnitude of C_4,

$$\frac{C_4}{\gamma_k^{\,L}N_k} \simeq \left[\frac{(k^2D)^{1/3}}{\max[\gamma_k, (k^2D)^{1/3}]}\right]^6. \qquad (15)$$

Thus, for $\gamma_k \gg (k^2D)^{1/3}$, we find that this term can be neglected and we recover the quasilinear result. On the other hand, for $\gamma_k \ll (k^2D)^{1/3}$ we find that this mode-mode coupling process contributes at the same order as the linear term. It can be shown that in this case all $2n$-wave resonant-coupling processes lead to contribution of the same order and that summing up these contributions leads to result (11). Once more, these conclusions hold only when inequalities (12) are fulfilled. Indeed, the result (15) is obtained when $(\sum\omega)$ is negligible as compared with γ_k in the domain of integration for k_1 and k_2 given by $|k_1 - k| \lesssim (k^2 D)^{1/3}/v(k)$, $|k_2 - k| \lesssim (k^2 D)^{1/3}/v(k)$.

These results show that, in the frame of weak-turbulence theory, the nonlinear behavior of strongly correlated resonant particles leads to a modification of the growth rate at least in a 1D plasma whenever inequalities (12) are fulfilled.

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