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Random-Field Singularities in Position-Space Renormalization-Group Transformations

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A close connection between Griffiths-Pearce "peculiarities" in position-space renormalization-group transformations and singularities in the free energies of systems with quenched random fields is pointed out. For a large class of weight functions the recurrence relations for the coupling constants of a nonrandom Ising system exhibit singularities where there is a phase transition in a related system with a random field.

A fundamental assumption in the renormalization-group approach to statistical mechanics is that the new coupling constants which characterize the system after a change in the length scale are nonsingular functions of the old coupling constants. Griffiths and Pearce[1] (GP) have argued that in position-space renormalization-group transformations for the Ising model this assumption is sometimes violated. Here it is pointed out that for a large class of weight functions the renormalization transformation for a nonrandom Ising system exhibits singularities similar to those in the free energy of a system in which quenched random fields[2] are present.

In position-space renormalization the Boltzmann factor transforms as

$$e^{H'(\tau)} = \sum_\sigma w(\tau,\sigma) e^{H(\sigma)}, \tag{1}$$

where the weight function $w(\tau,\sigma)$, which relates the new and old spin variables,[3] satisfies $\sum_\tau w(\tau, \sigma) = 1$. The Ising spin variables τ and σ take the values ± 1. Prior to the work of GP it was generally assumed that a suitable weight function eliminates long-wavelength critical fluctuations from the sum in Eq. (1), making the right-hand side a nonsingular function of the coupling constants in H.

That there are infinitely many τ configurations for which the assumption of no singularities is violated may readily be seen for the model-I weight function of GP,

$$w_I(\tau,\sigma) = \prod_i \exp[(p\tau_i\sigma_i)/2\cosh p] \tag{2}$$

which associates a τ spin with each σ spin. Combining (1) and (2) gives a transformation which does not reduce the number of spins but which is particularly simple to analyze. The results can be readily generalized to a large class of weight functions which do eliminate spins.

With use of the model-I weight function, Eq. (1) becomes

$$H'(\tau) = N' f_p(\tau) = \ln \sum_\sigma \exp[p\sum_i \tau_i\sigma_i + H(\sigma) - N' \ln(2\cosh p)]. \tag{3}$$

It will be assumed that $H'(\tau)$ as defined by Eq. (3) may be expanded in a suitable set of interactions with a well-defined thermodynamic limit. However, GP were only able to prove these properties for a sufficiently large magnetic field in H. The quantity $f_p(\tau)$ in Eq. (3) clearly represents the free energy per spin of an Ising system with a local field $p_i = p\tau_i$ acting on site i. If all the τ spins point up, $f_p(\tau)$

is proportional to the free energy of the system with Hamiltonian $H(\sigma)+p\sum_i \sigma_i$ and has the same singularities. If the τ spins are ordered antiferromagnetically, $f_p(\tau)$ has the singularities associated with a staggered rather than a uniform field. Other τ configurations obviously lead to other hypersurfaces of singularities.

GP discuss the singularities of the coupling constants in an expansion of $H'(\tau)$ in terms of lattice-gas occupation numbers. In this Letter the singularity structure of the Ising coupling constants is considered. The Ising couplings are defined by the expansion

$$H'(\tau)=N'K_0'+\sum_i K_i'\tau_i+\sum_{i<j} K_{ij}'\tau_i\tau_j+\sum_{i<j<k} K_{ijk}'\tau_i\tau_j\tau_k+\dots\,. \tag{4}$$

The coupling constants may be calculated from $H'(\tau)$ with the obvious inversion formula[3]

$$\begin{pmatrix} N'K_0' \\ K_i' \\ K_{ij}' \\ K_{ijk}' \\ \cdots \end{pmatrix}=\frac{1}{2^{N'}}\sum_\tau \begin{pmatrix} 1 \\ \tau_i \\ \tau_i\tau_j \\ \tau_i\tau_j\tau_k \\ \cdots \end{pmatrix} H'(\tau). \tag{5}$$

Assuming for simplicity that H' and H only contain interactions between even numbers of spins and replacing H' by $N'f_p$ in Eq. (5), one finds

$$K_0'=[f_p]_{\rm av} \tag{6}$$

$$K_{ij}'=\tfrac{1}{2}N'[f_p(\delta_{p_i,p_i}-\delta_{p_i,-p_j})]_{\rm av} \tag{7}$$

and similar formulas for the other even couplings. The square brackets $[\;]_{\rm av}$ indicate an average over the fields $p_i=p\tau_i$. The Kronecker δ functions impose correlations on the fields at sites i and j.

An implicit assumption in most of the approximate position-space renormalization methods is that the short-range couplings are the most important. In the limit of large separation between sites i and j, it follows from Eq. (7) that K_{ij}' is proportional to the spin-spin correlation function $[\langle\sigma_i\sigma_j\rangle]_{\rm av}$ for a system with Hamiltonian H in a random field $\pm p$. For large separations, K_{ij}' should decrease rapidly with increasing p since the correlations set up by the parallel and antiparallel fields at sites i and j are destroyed by the strong random fields on the other sites. In the limit $p\to\infty$ the model-I transformation reduces to the identity transformation $H'=H$. One trivially sees that no new couplings are generated and that the transformation is nonsingular.

Equation (6) implies that the constant contribution K_0' to the free energy is singular where the random-field free energy is singular. One anticipates an $O(N^{-1})$ correction to the free energy upon inclusion of the correlations between the quenched fields as in Eq. (7) but not a change in the location of the singularities. Thus one can argue that in the thermodynamic limit all of the interaction constants K' coupling finite numbers of spins are singular functions of the coupling constants in H where $[f_p]_{\rm av}$ is singular, i.e., where there is a phase transition in the quenched random-field problem.

GP find quite different results for the coupling constant φ_1' of the term linear in n_i in an expansion of $H'(\tau)$ in terms of the lattice-gas occupation numbers $n_i=\frac{1}{2}(1+\tau_i)$. There are singularities in φ_1' (and in the constants $\varphi_{ij}',\varphi_{ijk}',\dots$ which couple finite numbers of particles, as well) where the free energy associated with the Hamiltonian $H(\sigma)-p\sum_i\sigma_i$ is singular. In the τ configurations which determine these coupling constants, all but a finite number of spins point down, i.e., all except a finite number of the local fields $p_i=p\tau_i$ are parallel. Thus one finds the singularities associated with a uniform field rather than a random field.

In terms of the Ising couplings, φ_1' has the expansion

$$\varphi_1'=2[K_1'-\sum_{1<i}K_{1i}'+\sum_{1<i<j}K_{1ij}'-\dots]. \tag{8}$$

Equation (8) implies that the infinite-spin Ising couplings must be included in the sum to obtain the correct analytic properties of φ_1' in the thermodynamic limit. The terms coupling finite numbers of spins all exhibit singularities where the random-field free energy is singular.

In the case of ferromagnetic interactions the lattice-gas interaction constants φ', which couple finite numbers of particles, are nonsingular in the critical region of the system with Hamiltonian H. This also appears to be the case for the Ising constants K' coupling finite numbers of spins. Mean-field theory,[2] Monte Carlo simulations,[4] finite-lattice calculations,[5] and exact results for the infinite-range[6] and spherical models[7] all indicate that above the lower critical dimensionality[8] the introduction of a random field depresses the critical temperature. Thus the assumption of a nonsingular renormalization does not appear to be violated in the critical region of the system of interest.

Most of the results reported thus far can be extended to weight functions which reduce the number of spins. One such weight function is model II of GP. It is identical with model I except that τ spins are only introduced on a fraction of the σ sites; for example, every second site of a square lattice. Another weight function considered by GP is the model-III weight function

$$w_{\rm III}(\tau,\sigma)=\prod_i \frac{\exp[p\tau_i(\sigma_{i1}+\ldots+\sigma_{in})]}{2\cosh[p(\sigma_{i1}+\ldots+\sigma_{in})]} \qquad (9)$$

introduced by Kadanoff.[9] It associates a cell spin τ_i with a cell of n σ spins. In the limit $p\to\infty$, weight functions II and III reduce to the decimation[10] and majority-rule[3] weight functions, respectively.

For model II, $f_p(\tau)$ is given by Eq. (3) where the sum on i only includes those sites on which there are τ spins. For model III, $H(\sigma)-N'\ln(2\cosh p)$ in Eq. (3) is replaced by

$$H_{\rm eff}(\sigma)=H(\sigma)-\sum_i \ln\{2\cosh[p(\sigma_{i1}+\ldots+\sigma_{in})]\}.$$

The logarithmic terms in $H_{\rm eff}$ oppose a ferromagnetic ordering. Thus for a ferromagnetically coupled H and a fixed p one expects the system with Hamiltonian $H_{\rm eff}$ to undergo a paramagnetic-ferromagnetic transition at a lower critical temperature than the system with Hamiltonian H. For both weight functions it is clear that there are τ configurations (for example, all τ spins parallel) for which the right-hand side of Eq. (1) is singular.

In model II the random fields over which one averages in computing the K' only act on a fraction of the σ spins. In model III all of the spins in a given cell experience the same random field. Just as with model I, for a ferromagnetically coupled system one would expect singularities in the random-field free energy to occur at a lower critical temperature than that corresponding to H, if the singularities occur at all.[11]

GP first pointed out that for models I–III $H'(\tau)$ is in general a singular function of the coupling constants in $H(\sigma)$. From the study reported here it is clear that the location of the singularities in the recrusion relation for the coupling constants depends on the set of interactions considered. The lattice-gas interaction constants φ' coupling finite numbers of particles and the Ising constants K' coupling finite numbers of spins have constant and random-field singularities, respectively. In the τ configurations which determine the φ', all except a finite cluster of spins are parallel. The K' are determined by averaging over all τ' configurations. Neither the φ' nor the K' coupling finite numbers of particles are singular in the critical region of H.

Equation (7) implies that with increasing separation K_{ij}' has the same characteristic decay as the spin-spin correlation function in a random magnetic field. (The corresponding correlation function for φ_{ij}' involves a constant field p.) Since the most common approximation methods retain only a few short-range couplings, it is disconcerting to realize that the infinite-spin couplings must be included in formulas such as (8) to obtain the correct analytic properties of the quantity on the left.

According to domain-wall arguments[8] the lower critical dimensionality for a phase transition in an Ising ferromagnet in a random field is $d=2$. Thus, for $d>2$, random-field singularities can be expected in the renormalization transformation. It is unclear at present whether there is a phase transition for $d=2$.

In closing, I repeat that the results reported here are somewhat speculative because of the nonrigorous discussion of the thermodynamic limit.

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