Instability of Persistent Currents and Heat Flow in Superfluid ^{3}He-A

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A Landau expansion of the normal-mode amplitude shows that the instability of helical textures in a magnetic field parallel to the superflow always occurs as an inverted bifurcation that does not represent a transition to a nearby steady configuration. Above a critical magnetic field, for example, physical arguments show that a persistent current in a torus will be partially quenched, leading to a stable wide-angle helix, but that the corresponding texture in an applied heat flow will oscillate anharmonically.

In the presence of relative superflow with $\vec{v}_{sn} = \vec{v}_s - \vec{v}_n$, the hydrodynamic theory of ^{3}He-A predicts a spontaneous distortion[1] of a uniform configuration into a helical texture.[2,3] This second-order Landau transition is very similar to the onset of Bénard convection rolls[4,5]; it occurs at a critical value of an external stress such a parallel magnetic field or a decreased temperature. The apex angle of the helix grows with increasing stress until the helix itself becomes dynamically unstable. The present work examines this new transition in the dipole-locked regime with a Landau expansion for the time derivative of the unstable mode. The transition always occurs as an inverted bifurcation that produces a catastrophic deformation, rather than another second-order transition to a new distorted state. We study two specific models for the relative $\vec{v}_{sn}$: external heat and superflow in a torus. Both systems support stable helices but behave quite differently beyond the threshold of instability.

In the dipole-locked hydrodynamic approximation, the order parameter of ^{3}He-A is a complex orbital vector $\hat{m} + i\hat{n}$. Equivalently, it may be characterized by the unit vector $\hat{l} = \hat{m} \times \hat{n}$ and the phase Φ associated with rotations about $\hat{l}$. The basic hydrodynamic equations[6] are the conservation of mass and momentum, and the dynamics of $\hat{l}$ and Φ, which just suffice to determine the seven hydrodynamic variables $\vec{v}_n$, p, $\hat{l}$, and Φ. For simplicity, we here fix $\vec{v}_n$ externally and treat the fluid as incompressible. The pressure p will then be determined by the phase equation.

For helical textures, it is natural to use the Euler angles[3,7,8] (α,β,γ) of the triad $\hat{l},\hat{m},\hat{n}$ relative to some standard configuration with $\hat{z}$ along $\vec{v}_n$. The superfluid velocity is given by $\vec{v}_s = -\cos\beta\nabla\alpha - \nabla\gamma$, where a factor of $\hbar/2m_3$ has been absorbed to give $\vec{v}_s$ the dimensions of a wave number. To simplify the problem, consider one-dimensional textures depending only on z and t. A direct evaluation of the hydrodynamic free-energy density[1] then gives

$$\begin{aligned} f = {} & \tfrac{1}{2}(\rho_s^{\parallel} + \rho_0 \sin^2\beta)(\alpha' \cos\beta + \gamma' + v_{nz})^2 \\ & - C_0\alpha' \sin^2\beta\cos\beta(\alpha'\cos\beta + \gamma' + v_{nz}) \\ & + \tfrac{1}{2}\sin^2\beta(K_b\cos^2\beta + K_t\sin^2\beta)\alpha'^2 \\ & + \tfrac{1}{2}(K_b\cos^2\beta + K_s\sin^2\beta)\beta'^2 + \tfrac{1}{2}\lambda_m H^2\cos^2\beta, \end{aligned} \tag{1}$$

where a prime denotes a derivative with respect to z and the coefficients are known from model calculations.[6,7] Equation (1) expresses f in terms of the Euler angles but, as a result of our assumptions, it contains neither α nor γ. In addition, current conservation implies that $j_{0z} = \partial f/\partial v_{sz} = -\partial f/\partial\gamma'$ is now a conserved quantity, which can be made explicit with a Legendre transformation $\tilde{f} = f - \gamma'\partial f/\partial\gamma'$, analogous to the Routhian of classical mechanics.[3,9] In terms of this new function, the dynamical equation for $\hat{l}$ becomes

$$\mu\sin^2\beta(\partial\alpha/\partial t + v_{nz}\alpha') = (\partial f/\partial\alpha')' - \partial\tilde{f}/\partial\alpha, \tag{2a}$$

$$\mu(\partial\beta/\partial t + v_{nz}\beta') = (\partial\tilde{f}/\partial\beta')' - \partial\tilde{f}/\partial\beta, \tag{2b}$$

where derivatives of $\tilde{f}$ are taken, with its remaining natural variables (α, β, j_{0z}) kept fixed.

A steady helix has a constant polar angle β_0 and a linearly increasing azimuthal angle $\alpha_0 = -u(z - v_n t)$. It is static in the frame with $\vec{v}_n = 0$ and otherwise precesses uniformly. Consider two ways to fix $\vec{v}_n$:

(1) *Superflow in a torus.*—The rotating walls lock the normal fluid, and we simulate the multiply connected geometry by applying periodic boundary conditions over a length L. Single-valuedness of the order parameter quantizes the "winding numbers" n_α and n_γ of the angles α and γ. When $\cos\beta_0 = \pm 1$, only $n_\alpha \pm n_\gamma$ is quantized, with $\alpha' \pm \gamma'$ an integral multiple of $2\pi/L$; thus the original uniform texture has quantized relative superflow $w = -(\gamma' \pm \alpha' + v_n)$. When the helix first appears, n_α and n_γ become separately quantized, fixing $\alpha_0' = -u$ and $\gamma_0' = s \equiv -(w \pm u + v_n)$. The corresponding relative velocity $v_{snz} = w + u(\cos\beta_0 \pm 1)$

is no longer quantized; it depends on β_0, which is determined as a function of u and s through the equation $(\partial f/\partial \beta)_0 = 0$. With increasing external stress, a small-angle solution first occurs when[8,10]

$$\lambda_m H^2 + w^2[(C_0+\tfrac{1}{2}\rho_s^{\parallel})^2/K_b - \rho_0] > K_b(u-u_h)^2, \quad (3)$$

with $u_h = (C_0 + \frac{1}{2}\rho_s^{\parallel})w/K_b$. Equation (3) is first satisfied when $u = u_h$, giving a characteristic pitch proportional to the original relative superflow w.

(2) *Heat flow.*—The condition of zero mass flow in the laboratory frame requires $j_{0z} = -\rho v_n$, where ρ is the total density. Since the current is fixed, it is natural to use the transformed free energy $\tilde{f}$, and β_0 is determined by the equation $(\partial\tilde{f}/\partial\beta)_0 = 0$. Small-angle solutions to this equation exist whenever (3) is satisfied with w replaced by $j_{0z}/\rho_s^{\parallel}$. The onset of helical textures is the same in heat flow and the torus, but the evolution of the textures under increasing stress differs markedly because of the difference between the fixed quantities (j_{0z} in heat flow, n_α and n_γ in a torus).

To study the stability of the helices, we use Eq. (2) for both heat flow and flow in a torus, noting that the current j_{0z} in a torus depends on the equilibrium value of β_0, with the corresponding $\tilde{f}$ a function not only of β, but also of β_0 and u. If $\alpha = \alpha_0 + \delta\alpha$ and $\beta = \beta_0 + \delta\beta$, small deformations $\propto e^{ikz}$ about the equilibrium helix are stable for all k if[3] $ac - b^2 > 0$, where

$$a \equiv \frac{1}{\sin^2\beta_0}\left(\frac{\partial^2\tilde{f}}{\partial\alpha'^2}\right)_0, \quad b \equiv \frac{1}{\sin\beta_0}\left(\frac{\partial^2\tilde{f}}{\partial\alpha'\partial\beta}\right)_0, \quad c = \left(\frac{\partial^2\tilde{f}}{\partial\beta^2}\right)_0. \quad (4)$$

For small-angle helices in the presence of heat flow, this condition is more restrictive[11] than Eq. (3) by a factor of 3 on the right-hand side, so that changing the heat flow (and thus u_h) can induce an instability. A similar condition holds for a torus. Larger-angle helices require numerical work. If the helix is formed with $u = u_h$ by increasing the magnetic field near T_c in the presence of a fixed w (in a torus) or fixed j_{0z} (for heat flow), the helix becomes unstable when

$$\beta_0 \approx 0.599, \quad \lambda_m H_c^2 \approx 0.277\, j_{0z}^2/\rho_s^{\parallel} \text{ (heat flow)}; \quad (5a)$$

$$\beta_0 \approx 0.622, \quad \lambda_m H_c^2 \approx 0.289\rho_s^{\parallel} w^2 \text{ (torus)}. \quad (5b)$$

For the typical value $v_{sn} = 0.01$ cm/sec in the uniform state, the instability occurs at $H_c \approx 1.5$ Oe.

A quantitative analysis of the behavior beyond the instability threshold starts from Eqs. (2). We first study the linearized equations of motion writing, for example, $\delta\alpha(z,t) = \sum_m \delta\alpha_m(t)\, e^{imk_0 z}$, where $k_0 = 2\pi/L$ and $\delta\alpha_{-m}^* = \delta\alpha_m$ to ensure reality. The associated normal modes ξ_m and η_m are linear combinations that satisfy the equations of motion

$$\dot{\xi}_m + ik_0 m v_{nz}\xi_m = \sigma_m^{(+)}\xi_m, \quad \dot{\eta}_m + ik_0 m v_{nz}\eta_m = \sigma_m^{(-)}\eta_m; \quad (6)$$

$$\mu\sigma_m^{(\pm)} = -\tfrac{1}{2}[c + m^2k_0^2(d+a)] \pm\{\tfrac{1}{4}[c + m^2k_0^2(d+a)]^2 + m^2k_0^2[b^2 - ac - m^2k_0^2 ad]\}^{1/2}. \quad (7)$$

[see Eq. (4)] and $d = (\partial^2 f/\partial\beta'^2)_0$. Evidently, $\sigma_m^{(-)}$ is always negative, and the η modes are never critical. On the other hand, if $k_0^2 ad < b^2 - ac < 4k_0^2 ad$, the mode ξ_1 exhibits exponential growth, but those with $m \geq 2$ remain stable. Note the importance of the finite length L, for otherwise many ξ modes would become unstable together.

To analyze the ensuing nonlinear behavior, we expand Eqs. (2) through third order in δa and $\delta\beta$, assuming that $\xi_1(t) = A\exp(-ik_0 v_n t)$, where A is a real time-dependent amplitude. A detailed examination shows that the modes $m=0$ and $m=2$ are of order A^2, and that the originally orthogonal mode $\eta_1(t)$ is of order A^3. To leading order in the small parameter $k_0 a/b = O(\hbar k_0/2m_3 v_{sn})$, only the modes ξ_1 and ξ_2 contribute through order A^3, and we find the approximate Landau equation

$$\mu\dot{A} = \mu\sigma^{(+)}A + (k_0^2/6d)(3e - 3fa/b + ga^2/b^2)^2 A^3, \quad (8)$$

where e, f, and g are various third derivatives of $\tilde{f}$. This equation is our first basic result, for the coefficient of A^3 is *positive definite* independent of the details of Eq. (1). Thus the instability of the helical texture always appears as an *inverted bifurcation*,[5] with $A(t)$ increasing catastrophically for $\sigma^{(+)} > 0$. If the collapse of the helical texture occurs with increasing magnetic field at fixed $u = u_h$, we find $3e - 3fa/b + ga^2/b^2 \approx -1.077\rho_s^{\parallel}$ for heat flow and $-0.8039\rho_s^{\parallel}$ for a torus; helices in heat flow experience more rapid growth beyond threshold.

Can these phenomena be observed? For $k_0 \to 0$, the growth rate $\sigma_1^{(+)}$ first becomes positive when $b^2 \approx ac$, at the critical field given by Eqs. (5). Since the condition $\sigma_1^{(+)} = 0$ defines H_c, the growth rate $\sigma_1^{(+)}$ increases linearly from zero for small positive $H - H_c$. When $b^2 - ac$ reaches $4k_0^2 da$, the mode ξ_2 also becomes critical, and $\mu\sigma_1^{(+)}$ has the approximate value $3k_0^4da/c$. Using the known value[6] of μ near T_c, we estimate the corresponding growth rate $\sigma_1^{(+)} \approx 3.8\times10^{-3}\, k_B T_c/\Delta$ sec^{-1} for $v_{sn} \approx 0.01$ cm/sec and $L \approx 1$ cm. The associated increase in H should be $H - H_c \approx 0.2$ Oe. Within this limited range, it may be feasible to verify experimentally that the linearized growth rate $\sigma_1^{(+)}$ is indeed proportional to $H - H_c$.

What happens once the system leaves the regime of the above nonlinear analysis? We suggest that the inverted bifurcation signals the onset of intrinsically time-dependent dissipative behavior. The free energy of the helical states as a function of the opening angle β_0 provides qualitative insight into the possible character of the textures.

In a torus, the pitch of the helix $u_h = -\alpha_0'$ and the quantity $s = w - u_h + v_n$ are fixed; the appropriate free energy is given by f in Eq. (1), which has α, β, and γ as its natural variables. In heat flow the helices have u_h and j_{0z} fixed, and the appropriate free energy is $\tilde{f}$, with α, β, and j_{0z} as its natural variables. In either case, the helical solutions represent a local minimum of the appropriate free energy. Curve a of Fig. 1 shows schematically the structure of $f(\beta_0)$ near $H \approx H_c$ (the depth of the local minimum has been exaggerated), and $\tilde{f}(\beta_0)$ is practically identical. The additional degrees of freedom associated with α and γ allow the system to escape from the local minimum at β_0 at H_c. The form of the free energy in curve a suggests that the texture will become time dependent, with the dynamical equations tending to move the system inexorably[1] toward $\beta = \pi$. At $\beta = \pi$, however, the two systems are likely to behave very differently.

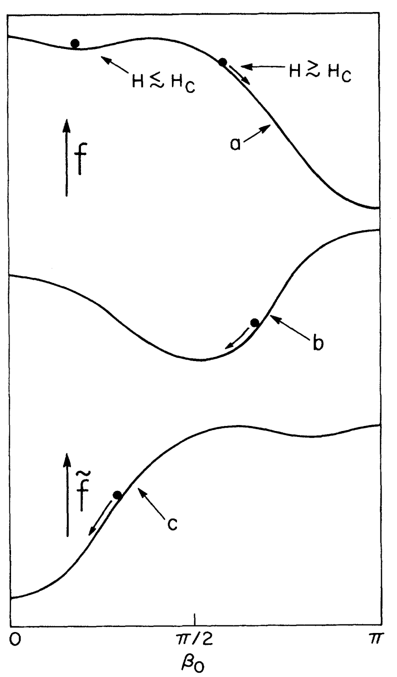

FIG. 1. Free-energy density for helical texture, as a function of opening angle. Curve a shows the free energy at the critical magnetic field where the helical textures become dynamically unstable. Curves b and c show, respectively, the appropriate free energies in a toroidal persistent current and heat flow after the opening angle β reaches π.

When β reaches π in the torus, the relative velocity is $v_{snz} = w - 2u_h \equiv \bar{w}$. Since $u_h = 0.6w$ in the dipole-locked limit near T_c, we have $\bar{w} = -0.2w$, and the direction of the relative velocity has actually *reversed*! The system is now in a uniform state with the relative velocity $\bar{w}$ parallel to $\hat{l}$ and magnetic field $H \approx H_c$. Furthermore, the degenerate angles α and γ are free to change, as long as the winding number of the *difference* $\alpha - \gamma$ remains quantized. To test for the stability of this uniform state with $\beta = \pi$, imagine it to fluctuate in a superposition of component helical states with all possible values of the pitch u. In the dipole-locked regime, the dynamical equations predict a maximum growth rate for a helix with pitch $\bar{u}_h = -(C_0 + \frac{1}{2}\rho_s^{\parallel})\bar{w}/K_b$. Replacing w by $\bar{w}$ and u_h by $\bar{u}_h$ in Eq. (1) with $H > H_c$ changes $f(\beta)$, as shown with curve b. It has a local maximum at $\beta = \pi$ and a new local *minimum* near $\beta = \pi/2$ that turns out to be dynamically *stable*. Presumably the system will relax to this minimum leaving a steady helix with pitch $\bar{u}_h$. Thus the quenching of the persistent supercurrent is incomplete, because the factor $1 - 2u_h/w$ in the relative velocity $\bar{w}$ allows the magnetic field to dominate the texture, preventing the process from repeating.

When β reaches π in heat flow, the pitch of the helix is again free to change, but now the total current must remain fixed, in contrast to the flow reversal in a torus. The uniform texture $\beta = \pi$ is also unstable in the presence of heat flow, and the maximum growth rate occurs for a helical fluctuation with $u = -u_h$. The free energy $\tilde{f}(\beta)$ with $u = -u_h$ (curve c) is just a reflection about $\beta = \pi/2$ of $\tilde{f}(\beta)$ with $u = u_h$ (curve a). The system evolves by decreasing β, escaping from the local minimum because $H > H_c$ and continuing toward $\beta = 0$. At $\beta = 0$ the value of u changes from $-u_h$ to u_h and the process repeats. Thus β oscillates between 0 and π, producing a time-dependent state whose motion is anharmonic, but periodic.[12] Hook and Hall[13] have obtained similar phenomena by numerically integrating the dynamical equations. They include the effects of boundaries, which increases the difficulty and may explain part of the complicated motion they report. Still, the mechanism seems clear: The system evolves toward a local minimum of $\tilde{f}$ at $\beta=0$ or π; upon reaching the local minimum, $\tilde{f}$ changes, with the local minimum now a local maximum. The external source of the heat current provides the energy dissipated by this process.

The preceding analysis, based on helical solutions, cannot be quantitatively correct, for it is likely that the unstable helices develop into more complicated time-dependent states. Nevertheless, we expect the qualitative behavior to confirm our second basic result that an applied parallel magnetic field $H>H_c$ should induce a marked time-dependent deformation, whose character depends on the nature of the experiment: A persistent current in a torus should lead to a stable wide-angle helix with reversed but diminished supercurrent, whereas heat flow should produce anharmonic but periodic oscillations of the texture.

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Nematic-Isotropic Transition in Liquid Crystals

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Correlation functions and the Cotton-Mouton coefficient are calculated for liquid crystals beyond the mean-field approximation. My results in the context of a first-order transition are compared with the recent experiments of Keyes and Shane for N-[*p*-methoxybenzylidine]-*p*-butylaniline (MBBA) connecting with the possible tricritical nature of the nematic-isotropic transition.

Recently, Keyes and Shane[1] measured the gap exponent Δ for the nematic-isotropic (N-I) phase transition in N-[*p*-methoxybenzylidine]-*p*-butylaniline (MBBA) in the isotropic phase. They found $\Delta=1.26+0.10$ which is consistent with the tricritical value $\Delta=1.25$ but differs from the mean-field prediction $\Delta=2$, giving the impression that the N-I transition is actually tricritical in nature. In this Letter, among other things, we show that by going beyond the mean-field approximation the so-called gap exponent Δ is not a constant but in general a function of temperature T. Depending on the temperature range under consideration, the effective exponent can deviate from the mean-field value and may be equal to 1.59, for example. Therefore, the measurement of Δ alone is insufficient in determining the critical or tricritical nature of the N-I transition. In addition, the deviation of the inverse of the Cotton-Mouton coefficient from linearity just above T_c is explained.

It has been known for some time that the de Gennes-Landau theory[2] is inapplicable near T_c in the isotropic phase. More recently, contrary to the current belief,[3] Lin and Cai[4] have shown that, quantitatively speaking, the same

