Particle Diffusion by Magnetic Perturbations of Axisymmetric Geometries

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The quasilinear theory of collisionless test particle diffusion in stochastic magnetic fields is extended to include the effects of finite gyroradius, particle drifts, and magnetic trapping. Runaway-electron confinement is substantially improved relative to earlier estimates which assumed that particles exactly followed field lines. Trapped particles are not expected to be stochastic.

We consider the effects of finite gyroradius, particle drifts, and magnetic trapping on particle diffusion due to magnetic perturbations of axisymmetric toroidal equilibria. Previous authors[1,2] have considered the approximation in which particles exactly follow stochastic magnetic field lines. The predicted transport rate was very large even for modest perturbation levels. Here, we demonstrate quantitatively that the more realistic particle motions in a torus may reduce the stochastic transport substantially. In particular, we predict that runaway electrons are significantly better confined than the primitive theories indicate.

We consider two types of magnetic perturbations: those arising from microturbulence,[1] e.g., from drift or tearing modes, and those arising from a coherent magnetic "ripple" field, due either to coil errors or introduced explicitly as in ripple injection schemes.[3] We allow for both trapped and passing particles. In this Letter, we assume for explicitness that the particles are sufficiently far from the trapped-passing separatrix so that the rapid variation of the bounce frequency with bounce action in this region can be ignored. For passing particles with zero gyroradius, our results reduce correctly to the collisionless formulas given in Refs. 1 and 2.

The principal results are as follows[4]: (a) The diffusion of passing particles in turbulence is reduced by three effects. In order of importance, these are (i) an averaging over the mode profile due to guiding-center drifts, (ii) a shift due to drifts of the radius at which a particle is resonant with a given mode, and (iii) an averaging over the mode profile due to finite gyroradius. (b) Trapped particles in turbulence are not expected to be stochastic for reasonable turbulence levels. (c) In a ripple field, passing particles which are not too far from the separatrix can be stochastic for perturbation fields of strength exceeded by proposed ripple injection schemes.

We use a canonical formalism, which deals succinctly with the unperturbed motion and isolates the resonances due to the perturbation simply and explicitly. The variables parametrizing phase space are the canonical momenta $\vec{I}$ which are constants of the unperturbed motion, and their conjugate coordinates $\vec{\Theta}$. We deal only with axisymmetric equilibria, for which one may take $\vec{I} \equiv (\mu, J_b, p_\varphi)$ and $\Theta \equiv (\Theta_g, \Theta_b, \Phi)$. Here μ is mc/e times the usual magnetic moment; the conjugate coordinate is Θ_g, the gyrophase. Conjugate to the bounce action J_b is the bounce phase Θ_b. The third action is p_φ, the canonical angular momentum. Since p_φ determines the flux surface about which a particle oscillates in the course of a bounce period, it is diffusion in p_φ which dominantly determines diffusion in the "radial" coordinate r (in general, a flux surface label). Conjugate to p_φ is Φ, the bounce-averaged value of the toroidal angle φ. Because $\vec{I}$ is constant in the absence of perturbations, the unperturbed motion is trivial: $\vec{\Theta}$ develops linearly in time, with frequency $\Omega(\vec{I}) \equiv (\Omega_g, \Omega_b, \Omega_\varphi)$. Here Ω_g is the bounce-averaged gyrofrequency, Ω_φ is the bounce-averaged toroidal drift, and Ω_b is the usual "bounce" frequency. (The concept of bounce motion applies to passing as well as to trapped particles. For the former, the bounce period $2\pi/\Omega_b$ is approximately $2\pi qR/v_\parallel$, the time for a passing particle to traverse a connection length.)

The theory consists of two parts. (1) For stochastic motion to occur, the perturbation strength must be sufficiently large that a stochasticity threshold, determined by an appropriate resonance overlap criterion, is reached. (2) Once this occurs, the stochastic motion of a particle in $\vec{I}$ space is determined by a diffusion tensor $\vec{D}(\vec{I})$ in that space. Both the threshold and $\vec{D}$ depend on the field-particle coupling coefficients $g_{\vec{l}}$, the Fourier amplitudes of the perturbation Hamiltonian with respect to the angle variables.

To determine $g_{\vec{l}}$, we find it convenient to introduce the vector potential $\vec{A}$ and to work in the radiation gauge (scalar potential vanishes). If we introduce the notation $z \equiv (\vec{\Theta}, \vec{I})$, the perturbation Hamiltonian H_1 becomes

$$H_1(z,t) = -e\vec{v}(z)\cdot\vec{A}(\vec{r}(z),t). \tag{1}$$

Fourier transforming in $\vec{\Theta}$, one finds

$$H_1(z,t) = \sum_a \sum_{\vec{l}} g_{\vec{l}}(\vec{I},a)\exp(i\vec{l}\cdot\vec{\Theta} - i\omega_a t), \tag{2}$$

$$g_{\vec{l}}(\vec{I},a) = -(2\pi)^{-3}\int d^3\Theta \exp(-i\vec{l}\cdot\vec{\Theta})e\vec{v}(z)\cdot\vec{A}^a(\vec{r}(z),\omega_a). \tag{3}$$

Here a represents a set of mode numbers (e.g., the poloidal, toroidal, and radial quantum numbers) labeling the components $\vec{A}^a$ of the perturbing field. Time dependence $\vec{A}^a \sim \exp(-i\omega_a t)$ is assumed. In $\vec{l} \equiv (l_s, l_b, l_\varphi)$, each l takes on any integral value. By definition, the ripple field contains a single component a with $\omega_a = 0$, while for turbulence we assume a broad spectrum of modes with ω_a of order the drift frequency ω_*. In both cases, $\omega_a \ll \Omega_g$ for all species $s=i$ (ions), $s=e$ (thermal electrons), and $s=r$ (runaway electrons); so we can neglect effects from all gyroharmonics except $l_g = 0$. This implies that μ is conserved.

From a quasilinear analysis using the forms (2) and (3), Kaufman has shown that $\vec{D}$ is given by[5]

$$\vec{D}(\vec{I}) = \pi\sum_a\sum_{\vec{l}} |g_{\vec{l}}(\vec{I},a)|^2 \vec{l}\vec{l}\,\delta(\omega_a - \vec{l}\cdot\vec{\Omega}). \tag{4}$$

If interpreted literally, this expression is singular at each of the wave-particle resonances, and zero elsewhere. However, the nonvanishing Kolmogorov entropy τ_K^{-1} in the stochastic state and the consequent nonlinear mixing of orbits ensures that the resonances are smoothed, so that the sum over $\vec{l}$ is to be interpreted as a suitable integral as discussed in Ref. 1. The resulting well-known quasilinear expression is valid for autocorrelation time short compared to τ_K, so that τ_K does not appear explicitly in the final answer. This regime is reasonable for most turbulence levels of interest.

From Eq. (4), one reads off the resonance condition $\omega_a = \vec{l}\cdot\vec{\Omega}$. The stochasticity criterion is determined by considering the excursion $\Delta\vec{I}_{\vec{l}}$ or $\Delta\vec{\Omega}_{\vec{l}}$ in $\vec{I}$ or $\vec{\Omega}$ space induced by a single component $(\vec{l},a)$ of H_1 with which a particle is nearly resonant. Stochasticity ensues when this excursion is large enough to move the particle to the next resonance, i.e., when islands overlap. From Hamilton's equations, one finds

$$\Delta\vec{I}_{\vec{l}} = \vec{l}\,|2g_{\vec{l}}/\omega_{\vec{l}}|, \quad \Delta\vec{\Omega}_{\vec{l}} = (\partial\vec{\Omega}/\partial\vec{I})\cdot\Delta\vec{I}_{\vec{l}}, \tag{5}$$

where $\omega_{\vec{l}}$ is the trapping frequency of a particle in the potential well $[\sim\cos(\vec{l}\cdot\vec{\Theta} - \omega_a t)]$ of the selected perturbation component: $\omega_{\vec{l}} = \vec{l}\cdot\Delta\vec{\Omega}_{\vec{l}}$.

For turbulence, the particle resonantes with successive modes a localized at radii r_a with mode spacing $\delta \sim \rho_i/m$, where ρ_i is the ion gyroradius and m is the poloidal mode number, $m \sim \rho_i/r$. In this case, the stochasticity criterion is

$$1 < (\Delta r_{\vec{l}}/\delta)^2, \tag{6}$$

where $\Delta r_{\vec{l}} \equiv (\partial r/\partial p_\varphi)\Delta p_{\varphi\vec{l}}$ is the radial excursion due to component $(\vec{l},a)$.

For ripple, only a single component a is present and the resonance spacing $\Omega_\varphi \Delta l_\varphi$ in the l_φ direction is given by the fundamental of the perturbation, $\Delta l_\varphi = n_0 \sim 10$–$20$. This spacing is wider than the spacing $\Omega_b\Delta l_b = \Omega_b$ for the l_b direction. Thus, a particle moves along a chain of successive resonances $0 = \vec{l}'\cdot\vec{\Omega}(r_{l'})$, where $\vec{l}' = \vec{l}$, $\vec{l}\pm\hat{b}$ $\vec{l}\pm 2\hat{b}, \ldots$, with $\hat{b}$ a unit vector picking out the component l_b of $\vec{l}$. This yields the overlap criterion for ripple,

$$1 < (\vec{l}\cdot\Delta\vec{\Omega}_{\vec{l}}/\Omega_b)^2 \equiv (\omega_{\vec{l}}/\Omega_b)^2. \tag{7}$$

One sees from Eqs. (4) and (5) that the coupling coefficients $g_{\vec{l}}$ play a central role in both parts of the theory. Expression (3) contains complete information both about the particle trajectory [through $r(z)$ and $v(z)$] and the mode $\vec{A}^a(\vec{r},\omega_a)$, and therefore permits as much realism to be put into the theory as is desired. We write $\vec{r} = \vec{R} + \vec{\rho}$, where $\vec{R}$ is the guiding-center position and $\vec{\rho}$ describes the gryomotion. Sufficiently far from the separatrix, one can write $\vec{R}$ for both trapped and passing particles in the form

$$\vec{R}(\vec{\Theta}) = \hat{r}(r_b + r_1\cos\Theta_b) + \hat{\Theta}(\theta_0\Theta_b + \theta_1\sin\Theta_b) + \hat{\varphi}(\Phi + \varphi_1\sin\Theta_b). \tag{8}$$

Here $\theta_0 = 0$ for trapped particles and $\theta_0 = 1$ for passing particles (thus describing the secular motion in θ of the latter). Parameter r_1 is the radial excursion due to drifts: For trapped particles $r_1 \sim q\rho\epsilon^{-1/2}$, the banana width; for passing particles $r_1 \sim q\rho$. Finally, θ_1 and φ_1 describe both drift motion normal to the field lines and in the flux surface, and that part $\delta v_\parallel$ which is oscillatory because of the μB well. When the latter effect dominates, one has $\varphi_1/\theta_1 \simeq q(r_b)$. For trapped particles, θ_1 is the poloidal angle at the turning point, and so for a typical particle is an appreciable fraction of π: $\theta_1 \sim 1.5$. For a passing particle, one may estimate θ_1 from the contribution due to motion along field lines, $\theta_1 \sim \delta v_\parallel/v_\parallel$, and from the contribution due to perpendicular drifts, $\theta_1 \sim r_1/r$.

Using Eq. (8) in Eq. (3), one finds approximately

$$g_{\vec{1}} = -eJ_0(k_\perp\rho)\sum_m \{(\Omega_b b_0 A_\theta + \Omega_\varphi A_\varphi)J_{l_b-\theta_0 m}(y_1) + \tfrac{1}{2}\Omega_b(\theta_1 A_\theta + \varphi_1 A_\varphi)[J_{l_b-\theta_0 m-1}(y_1) + J_{l_b-\theta_0 m+1}(y_1)]\}. \tag{9}$$

Here A_θ denotes the quantity

$$A_\theta{}^a(r, m, l_\varphi) \equiv (2\pi)^{-2}\int d\theta \int d\varphi \exp[-i(m\theta + l_\varphi\varphi)]A_\theta{}^a(r, \theta, \varphi),$$

averaged over width r_1 about $r = r_b$; y_1 is given by $y_1{}^2 \equiv (m\theta_1 + l_\varphi\varphi_1)^2 + (k_r r_1)^2$, where k_r is the dominant radial wave number of A_θ.

One recovers the results of Refs. 1 and 2 by considering passing particles ($\theta_0 = 1$) with the drifts "turned off" ($\theta_1 = \varphi_1 = r_1 = y_1 = 0$), taking $k_\perp\rho = 0$, and considering a turbulent spectrum. Then there is a single term $m = l_b$ in the m sum in Eq. (9), and $l_\varphi = -n$, so that $\sum_{\vec{1}} = \sum_{m,n}$. To examine radial transport, we take the component $D_{rr} \equiv (\partial r/\partial p_\varphi)^2 D_{p_\varphi p_\varphi}$ of the diffusion tensor, finding

$$D_{rr} = \sum_{m,n} (R\Omega_\varphi)^2 \mathcal{B}_{1,0}{}^2 \pi\delta(m\Omega_b - n\Omega_\varphi). \tag{10}$$

Here $\mathcal{B}_{1,0}$ is the ratio of $B_r{}^a(r_a)$ (the magnitude of the radial perturbation at the radius r_a at which it is a maximum) to the unperturbed field; this is the measure of perturbation strength used in Refs. 1 and 2. Recognizing that $\delta(m\Omega_b - n\Omega_\varphi) = \Omega_\varphi{}^{-1} \times \delta(m/q(r) - n)$, one sees that Eq. (10) is the same as the results of Refs. 1 and 2, multiplied by the factor $R\Omega_\varphi$ which converts from diffusion with change in position along a field line to diffusion with change in time.

To see how this result is modified by the new effects, we find it useful to define a measure of the effective perturbation strength by

$$\mathcal{B}_1 \equiv \mathcal{B}_{1,0} J_0(k_\perp\rho)J_{l_b-\theta_0 m}(y_1)\Gamma. \tag{11}$$

The factor $J_0(k_\perp\rho)$ is the modification of $\mathcal{B}_1/\mathcal{B}_{1,0}$ due to finite gyroradius. For turbulence, $k_\perp \sim \rho_i{}^{-1}$. In addition, the runaway perpendicular energy, though much less than its parallel energy, may be large enough[6] ($E_\perp \sim 50$ keV for $T_e \simeq 1$ keV, $n_e \simeq 10^{14}$ cm^{-3}) that ρ_r is an appreciable fraction of ρ_i. For purposes of estimation, therefore, we take $k_\perp\rho \sim 1$, $J_0(k_\perp\rho) \sim \frac{2}{3}$ for $s = i, r$. The factor $\Gamma \equiv B_r{}^a(r_{\text{res}})/B_r{}^a(r_a) \lesssim 1$ measures the fact that the radius $r_{\text{res}} = r_b$ at which a particle is resonant with mode a is displaced, in the presence of drifts, by a distance $\sim r_1$ from the radius r_a at which the mode has $k_\parallel = 0$. If the radial dependence were $B_r{}^a(r) \sim \exp[-(r - r_a)^2/w_a{}^2]$, one would have $\Gamma = \exp[-(r_1/w_a)^2]$, strongly dependent on r_1. For $s = i, r$, we estimate $\Gamma \sim \frac{1}{3}$.

Finally, the factor $J_{l_b-\theta_0 m}$ in Eq. (11) quantifies the modification of the field-particle coupling due to drifts and the modulation of $v_\parallel$ by the μB well. [This form is appropriate for passing particles; a similar factor arising from the last line of Eq. (9) enters for trapped particles.] For turbulence and $s = i, r$, one has $y_1 \sim 2$ or 3, $l_b = m$, so that for passing particles $J_{l_b-\theta_0 m}(y_1) = J_0(y_1) \sim \frac{1}{3}$. If we combine this with the effects discussed in the preceding paragraph, we estimate that $\mathcal{B}_1/\mathcal{B}_{1,0} \sim \frac{1}{13}$. Since $\overleftrightarrow{D} \sim g_{\vec{1}}{}^2 \sim \mathcal{B}_1{}^2$, one estimates that radial transport can be reduced from the estimates of Refs. 1 and 2 by more than two orders of magnitude. This may explain the anomalously long confinement times of runaway electrons referred to in Ref. 2.

Using Eq. (9), one may evaluate criterion (6):

$$1 < |(L_s/k_\theta\delta^2)\mathcal{B}_1|, \tag{12}$$

where L_s is the shear length. This is the same criterion as in Ref. 1 except that $\mathcal{B}_1$ replaces $\mathcal{B}_{1,0}$. Thus, the perturbation strength $\mathcal{B}_{1,0}$ must be about 13 times larger than the previous estimates in order to satisfy (12), i.e., roughly 2.5×10^{-6} instead of 2×10^{-7}.

For ripple, $\Gamma = J_0(k_\perp\rho) \simeq 1$ and $y_1 \simeq n\varphi_1 \simeq qn\theta_1 \sim 30$. Because $y_1 \gg 1$, there is a spread due to the particle's drift and $\delta v_\parallel$ motions, $\Delta l_b \sim 2y_1$, in the effective spectrum which a particle sees. This spread permits the coherent perturbation to induce stochastic motion. For passing particles $J_{l_b-m}(y_1) = J_{qn}(qn\theta_1) \sim (\theta_1)^{-qn}J_{qn}(qn)$, where $J_\nu(\nu) \sim \nu^{-1/3}$. Stochasticity is thus most easily induced on particles having $\theta_1 \sim 1$, for which $J_{qn}(y_1)$ is near its maximum. Evaluating (7), one finds

$$1 < |(q^3 R l_\varphi/\epsilon L_s)\mathcal{B}_1|. \tag{13}$$

For $\theta_1 \sim 1$, a perturbation strength $\mathcal{B}_{1,0} \gtrsim \frac{1}{50}$ is needed for stochasticity. Proposed ripple injection schemes[3] satisfy this.

The strength and coherent nature of the ripple perturbation make integration along unperturbed trajectories untenable for trapped particles; so we consider trapped particles only in turbulent spectra. In this case, (7) is more difficult to satisfy than (6), by a factor of roughly $m^2 \sim 10^4$. This requires a large turbulence level $\mathfrak{B}_{1,0} > 10^{-2}$, a regime which we do not consider here. Because (7) is not satisfied, a particle makes only a fraction of an oscillation in the $\cos(\vec{1}\cdot\vec{\Theta})$ well during the first half of its bounce period, then retraces its motion during the return half. Mathematically, this is manifested by a factor $J_{-1} + J_1 = 0$ ($l_b = 0$) which replaces the factor $J_{l_b - m}$ appearing in $\mathfrak{B}_1$. We conclude that trapped electrons should not be stochastic.

Finally, one may consider trapped ions in turbulence. Here, since $l_\varphi \Omega_\varphi / \omega_a \sim \epsilon \ll 1$, the resonance condition requires that $l_b \simeq \omega_a/\Omega_b \approx q\epsilon^{-3/2} \sim 12$. Thus the terms $J_{l_b \pm 1}$ in $\mathfrak{B}_1$ greatly reduce g_{Υ}, by a factor on the order of $(\epsilon^{3/2})^{l_b}$, and one expects no stochasticity for ions as well. The physical mechanism here is that, for $\omega_a \sim \omega_*$, the ions move too slowly to resonate with the waves.

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Observation of Anomalous Heat Capacity in Liquid ^{3}He near the Superfluid Transition

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The specific heat of liquid ^{3}He from 0.8 to 20 mK at zero pressure has been measured. Above ~ 3 mK the specific heat is linear in temperature and $C/nRT = 2.11\ \mathrm{K}^{-1}$, which is 30% less than the currently accepted value. Below 3 mK, C appears to deviate increasingly from this relationship reaching, at the superfluid transition $T_c = 1.04$ mK, a value 9% in excess of the extrapolated linear specific heat. This Letter discusses the anomalous behavior and its consequences with regard to the interpretation of our data.

There is an urgent need for precise specific-heat data on liquid ^{3}He in the vicinity of the superfluid transition over the whole pressure range. Some of the most fundamental tests of the current theories[1,2] on superfluidity in ^{3}He and of the Fermi-liquid theory can be carried out when accurate specific-heat data become available.

Below the superfluid transition the specific heat of liquid ^{3}He has been measured earlier by several groups.[3-8] The results, however, are not very consistent because of problems associated with thermometry and with background contributions to the heat capacity. The most reliable data seem to be those of Halperin *et al.*[5] along the melting curve. At low pressures no accurate data of the specific heat below 10 mK are available.

We report in this Letter measurements of the specific heat of liquid ^{3}He in the temperature range 0.8–20 mK at zero pressure. In order to be able to determine temperatures precisely we have developed a thermometer based on the magnetic susceptibility of cerous magnesium nitrate, diluted to 3% molar solution in the corresponding lanthanum salt (abbreviated as CLMN, cerium diluted in lanthanum magnesium nitrate).[9,10] In addition, by using a method of analysis based on the variation of the amount of liquid in the cell, we have been able to perform an accurate deter-

