SU(9) Grand Unification of Flavor with Three Generations

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A model based on the group SU(9) is proposed to relate three quark-lepton generations. There is sufficient unification to give two new relations between the nine fermion masses. The guiding principle is to minimize the number of Higgs scalars necessary to obtain mass relations.

Recently, there has been interest in the possibility of embedding several quark-lepton generations within a group that extends the minimal group[1] unifying color and flavor. The object is to gain insight into the number of generations, to relate the fermion masses, and eventually to study the Cabibbo-like mixing angles.

Along these lines, Georgi[2] has proposed a three-generation model based on SU(11), while one of us made an SU(7) model[3] which could accommodate any number of generations. Although the SU(7) model contains gauge bosons mediating interactions between different generations, the Yukawa couplings contain too many parameters to yield useful mass relations. Here we present an SU(9) model that remedies this defect, and makes some beginning in getting mass relations. The basic idea of the present Letter is to construct, systematically, a generalized grand unified model having a minimum number of Higgs scalars in order to predict mass relations.

The assumptions made are as in Ref. 3, but now we insist that the Yukawa couplings to Higgs scalars which generate the fermion masses have a minimal number of independent parameters. This imposes further restrictions on the possible choice of group rank and of the irreducible representations used for the fermion fields.

Suppose that we require that there are g generations of light fermions, each generation comprising fifteen left-handed helicity states transforming[1] as $\underline{5}^*\oplus\underline{10}$ under SU(5), together with the fifteen conjugate right-handed states. Then to minimize the Yukawa parameters we require that both the $g(\underline{10}^*)$ and the $g(\underline{5})$ each arise from a single irreducible representation of SU(N). To reduce further the number of independent mass parameters we require that the products $\underline{10}\otimes\underline{10}$ and $\underline{10}\otimes\underline{5}^*$ involved in generating the mass of $(u,c,t,\ldots)$ and $(e,d;\mu,s;\tau,b;\ldots)$ respectively, can couple to the *same* Higgs scalar irreducible representation of SU(N).

Suppose that the $g(\underline{10})$ arises from the representation $[N,m]$ (signifying the fundamental representation of SU(N) with m totally antisymmetrized tensor indices) and that the $g(\underline{5}^*)$ arises from $[N,m']$. Now in SU(5) the $(u,c,t,\ldots)$ mass terms arise from the couplings[1,4,5]

$$\bar{\Psi}(\underline{10}^*)_R\Psi(\underline{10})_L\varphi(\underline{5}\text{ or }\underline{45})+\bar{\Psi}(\underline{10})_L\Psi(\underline{10}^*)_R\varphi(\underline{5}^*\text{ or }\underline{45}^*),\tag{1}$$

whereas the $(e,d;\mu,s;\tau,b;\ldots)$ masses arise from

$$\bar{\Psi}(\underline{10})_L\Psi(\underline{5})_R\varphi(\underline{5}\text{ or }\underline{45})+\bar{\Psi}(\underline{5}^*)_L\Psi(\underline{10}^*)_R\varphi(\underline{5}\text{ or }\underline{45})+\bar{\Psi}(\underline{10}^*)_R\Psi(\underline{5}^*)_L\varphi(\underline{5}^*\text{ or }\underline{45}^*)$$
$$+\bar{\Psi}(\underline{5})_R\Psi(\underline{10})_L\varphi(\underline{5}^*\text{ or }\underline{45}^*).\tag{2}$$

Now, the $\underline{10}$'s and $\underline{5}^*$'s will be represented by spinors having indices $[\hat{\alpha}=(\alpha,a)$ where $\alpha=1-5$, $a=6-N]$ as follows:

$$\Psi_{\alpha\beta(m-2)}(\underline{10})_L;\quad \Psi_{\alpha\beta\gamma\delta(m'-4)}(\underline{5}^*)_L,\tag{3}$$

where $(m-2)$, $(m'-4)$ denote the number of latin indices. Thus the couplings in (1) are of the form

$$[\bar{\Psi}(\underline{10}^*)_R]^{\alpha\beta\gamma(N-m-3)}[\Psi(\underline{10})_L]_{\alpha\beta(m-2)}\varphi_{\gamma(N-2m-1)}(\underline{5}), \tag{4a}$$

$$[\bar{\Psi}(\underline{10}^*)_R]^{\alpha\beta\gamma(N-m-3)}[\Psi(\underline{10})_L]_{\alpha\delta(m-2)}\varphi_{\beta\gamma(N-2m-1)}{}^{\delta}(\underline{45}), \tag{4b}$$

$$[\bar{\Psi}(\underline{10})_L]^{\alpha\beta(m-2)}[\Psi(\underline{10}^*)_R]_{\alpha\beta\gamma(N-m-3)}\varphi^{\gamma(N-2m-1)}(\underline{5}^*), \tag{4c}$$

$$[\bar{\Psi}(\underline{10})_L]^{\alpha\delta(m-2)}[\Psi(\underline{10}^*)_R]_{\alpha\beta\gamma(N-m-3)}\varphi_{\delta}{}^{\beta\gamma(N-2m-1)}(\underline{45}^*). \tag{4d}$$

The couplings in (2) are correspondingly of the general form:

$$[\bar{\Psi}(\underline{10})_L]^{\alpha\beta(m-2)}[\Psi(\underline{5})_R]_{\alpha(N-m'-1)}\varphi_{\beta(m+m'-N-1)}(\underline{5}), \tag{5a}$$

$$[\bar{\Psi}(\underline{10}^*)_R]^{\alpha\beta\gamma(N-m-3)}[\Psi(\underline{5}^*)_L]_{\alpha\beta\gamma\delta(m'-4)}\varphi^{\delta(m+m'-N-1)}(\underline{5}^*), \tag{5b}$$

and similar other couplings. Now, for the same Higgs irreducible representation (irrep) to couple in both Eqs. (4) and (5) a necessary condition is that

$$N-2m-1=+(m+m'-N-1), \tag{6}$$

giving

$$3m+m'=2N. \tag{7}$$

This restriction will ensure the possibility that all the light quarks and leptons accept masses from the same Higgs irrep.

Equation (7) must be reconciled with our choices concerning the irreps for the fermions. In SU(6) or SU(7), no $[N,m]$ contains three or more $\underline{10}$'s or $\underline{5}^*$'s, and so no model with $g \geq 3$ can be made such that the above-mentioned restrictions can be satisfied. For example, the model of Ref. 3 gives no fermion mass relations beyond those already present in SU(5) itself.

In SU(8), one has $3(\underline{10})$ in $m=3$ or 4, and $3(\underline{5}^*)$ in $m'=5$ and 6, as can be seen from the decomposition into SU(5) irreps:

$$[8,m]\to\binom{3}{m}\underline{1}\oplus\binom{3}{m-1}\underline{5}\oplus\binom{3}{m-2}\underline{10}\oplus\binom{3}{m-3}\underline{10}^*\oplus\binom{3}{m-4}\underline{5}^*\oplus\binom{3}{m-5}\underline{1}, \tag{8}$$

where $\binom{a}{b}$ is the binomial coefficient $a![b!\times(a-b)!]^{-1}$. However, these m and m' values cannot satisfy the constraint, Eq. (7). Thus, we are led to consider SU(9). For $N=9$, $m=3$ and 5 each contain $4(\underline{10})$ while $m=4$ contains $6(\underline{10})$; similarly $m'=5$ and 7 contain $4(\underline{5}^*)$, $m'=6$ has $6(\underline{5}^*)$. To satisfy Eq. (7), one must choose $m=4$ and $m'=6$, thus allowing up to six generations.

In SU(10), we find that $m=3$ and 6 have $5(\underline{10})$, $m=4$ and 5 have $10(\underline{10})$; also $m'=5$ and 8 have $5(\underline{5}^*)$, $m'=6$ and 7 have $10(\underline{5}^*)$. For Eq. (7), one must employ $m=4$, $m'=8$ or $m=5$, $m'=5$; this limits the number of possible generations in SU(10) to five. In SU(11), the allowed choices are only $m=5$, $m'=7$ permitting up to twenty low-mass generations. Note that the model of Ref. 2 has $m=4$ and $m'=8$ or 9, thus requiring further independent Yukawa coupling parameters than we insist on here.

The smallest group satisfying our criteria is SU(9), and the next task is to ask how to minimize the total dimensionality of a model having $g=3$ generations (also the allowed values $g=4, 5, 6$). Let the SU(9) model contain the set of antisymmetrical fundamental representations $[m]$ according to

$$\{m\}=\oplus\sum c_M[M]. \tag{9}$$

If we define $b_i=c_i+c_{9-i}$ $(i=1-4)$ and $a_i=c_i-c_{9-i}$, the total dimensionality is

$$D=9b_1+36b_2+84b_3+126b_4, \tag{10}$$

and the no-anomaly requirement[6,7] gives

$$a_1+5a_2+9a_3+5a_4=0. \tag{11}$$

According to Georgi,[2] the number of light generations, g, is given by the difference between the number of $\underline{5}^*$'s and the number of $\underline{5}$'s and hence g fulfills

$$g=a_2+3a_3+2a_4. \tag{12}$$

To satisfy Eq. (7), we have the further constraint that $c_4, c_6 \geq 1$. To minimize D, one can explore small integer values of a_3 and a_4, to arrive at the following optimal models:

$$g=3, \quad D=420, \quad \{m\}=[2]\oplus[3]\oplus[4]\oplus[6]\oplus 10[8]; \tag{13}$$

$$g=4, \quad D=495, \quad \{m\}=2[3]\oplus[4]\oplus[6]\oplus[7]\oplus 9[8]; \tag{14}$$

$$g=5, \quad D=504, \quad \{m\}=2[3]\oplus[4]\oplus[6]\oplus 14[8]; \tag{15}$$

$$g=6, \quad D=585, \quad \{m\}=[2]\oplus 2[3]\oplus[4]\oplus[6]\oplus 19[8]. \tag{16}$$

In any of the above models, if we assume that in the SU(5) subgroup only the five-dimensional representation of Higgs contributes, the masses of the $(e,d;\mu,s;\tau,b;\ldots)$ arise from the couplings of the three types

$$\overline{\Psi}^{\alpha\beta ab}\Psi_{\alpha ab}\varphi_\beta, \tag{17a}$$

$$\overline{\Psi}^{\alpha\beta ab}\Psi_{\alpha ac}\varphi_{\beta b}{}^{c}, \tag{17b}$$

$$\overline{\Psi}^{\alpha\beta ab}\Psi_{\alpha cd}\varphi_{\beta ab}{}^{cd}. \tag{17c}$$

Similarly the $(u;c;t;\ldots)$ masses arise from one of the three generic forms

$$\overline{\Psi}^{\alpha\beta\gamma ab}\Psi_{\alpha\beta ab}\varphi_\gamma, \tag{18a}$$

$$\overline{\Psi}^{\alpha\beta\gamma ab}\Psi_{\alpha\beta ac}\varphi_{\gamma b}{}^{c}, \tag{18b}$$

$$\overline{\Psi}^{\alpha\beta\gamma ab}\Psi_{\alpha\beta cd}\varphi_{\gamma ab}{}^{cd}. \tag{18c}$$

In the case of the fundamental representation $\underline{9}$ for the Higgs [Eqs. (17a), (18a)] then putting ${}_0\langle\varphi_\gamma\rangle_0=\delta_{\gamma 5}V$ gives the unwanted mass degeneracies $m_e=m_\mu=m_\tau=\ldots=m_d=m_s=m_b=\ldots$ and $m_u=m_c=m_t=\ldots$. Using instead[8] the $\underline{315}$ of Higgs [Eqs. (17b), (18b)], we can set[9]

$${}_0\langle\varphi_{\gamma b}{}^{c}\rangle_0=\delta_{\gamma 5}\delta_b{}^{c}V_b, \tag{19}$$

for $b=6-9$. For $g=3$, one had the common mass ratios $m_u/m_d=m_c/m_s=m_t/m_b$ [in addition to the usual SU(5) relations] at the unification mass.[10] The remaining three generations contained in the SU(9) representations $\Psi_{\alpha\beta ab}$, $\Psi_{\alpha ab}$ presumably get superheavy masses; the simplest way to achieve this is to put one of the V_b's (say, V_9) superheavy. For $g=6$, the corresponding six mass ratios $(m_u/m_d=m_c/m_s=\ldots)$ are again equal but one has two additional mass relations, $m_6-m_5=m_2-m_1$ and $m_5-m_4=m_3-m_2$, between generations.

With the more general structure of Eqs. (17c), (18c), and with the vacuum value[9]

$$\langle\varphi_{\gamma ab}{}^{cd}\rangle=\delta_{\gamma 5}(\delta_{ca}\delta_{db}-\delta_{da}\delta_{cb})V_{ab}, \tag{20}$$

the relative masses of the generations are arbitrary, but we again keep equality of the six mass ratios $(m_u/m_d=m_c/m_s=\ldots)$.

These results for mass relations are unchanged if one uses, in Eqs. (17b), (18b) or (17c), (18c), the 45-dimensional representation of the SU(5) subgroup.

Of course, one would like to obtain absolute values for mass ratios between the generations, e.g., m_μ/m_e, but such a ratio either is predicted to be unity or appears as a (free) parameter characterizing the symmetry breaking SU(N) → SU(5) → SU(3)⊗SU(2)⊗U(1). Also, the choice of larger symmetry group SU(N), and of the fermion representations, seems to require stronger guidelines than are available; as stated before, supergravity may ultimately provide such guidelines.

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[8]Thus 315 is the minimum number of scalars for SU(9) to give useful mass relations. This number is at least tripled if one relaxes our contraints between m and m'.

[9]In writing Eqs. (19) and (20), we are assuming that a potential can be constructed having such an energy minimum. At the (classical) tree level there is a sufficient number of independent scalar self-couplings to achieve this.

[10]This mass formula obtained for $g=3$ is not unique to our particular model. What we consider noteworthy is that it is obtained here without any imposition of discrete symmetries on the Yukawa couplings. It thus emerges naturally and not only in the technical sense of that adverb.