Positivity of the Total Mass of a General Space-Time

Richard Schoen
Courant Institute of Mathematical Sciences, New York University, New York, New York 10003

and

Shing-Tung Yau
Department of Mathematics, Stanford University, Stanford, California 94305
(Received 16 July 1979)

The positive-mass conjecture states that for a nontrivial isolated physical system, the total energy, which includes contributions from both matter and gravitation, is positive. This assertion has been demonstrated in the important case when the space-time admits a maximal slice. Here this assumption is removed and the general mass conjecture is demonstrated. Also the topology of all possible asymptotically flat space-times satisfying the local energy condition is determined.

An initial data set for a space-time consists of a three-dimensional manifold N, a positive definite metric g_{ij}, a symmetric tensor h_{ij}, a local mass density μ, and a local current density J^i. The constraint equations which determine N to be a spacelike hypersurface in a space-time with second fundamental form h_{ij} are given by

$$\mu = \tfrac{1}{2}[R - \textstyle\sum_{i,j} h^{ij}h_{ij} + (\sum_i h_i{}^i)^2],$$

$$J^i = \textstyle\sum_j \nabla_j [h^{ij} - (\sum_k h_k{}^k) g^{ij}],$$

where R is the scalar curvature of the metric g_{ij}. As usual, we assume that μ and J^i obey the local energy condition,

$$\mu \geqslant (\textstyle\sum_i J^i J_i)^{1/2}.$$

An initial data set will be said to be asymptotically flat if for some compact set C, $N \backslash C$ consists of a finite number of components $N_1, \ldots, N_k$ such that each N_i is diffeomorphic to the complement of a compact set in R^3. Under such diffeomorphisms, the metric tensor will be required to be written in the form

$$g_{ij} = (1 + M/2r)^4 \delta_{ij} + p_{ij},$$

where

$$p_{ij} = O(1/r^2),$$

$$\nabla p_{ij} = O(1/r^3),$$

and

$$\nabla\nabla p_{ij} = O(1/r^4).$$

The components of h_{ij} will also be required to be $O(1/r^2)$.

The number M is called the ADM mass of N_i (see Arnowitt, Deser, and Misner[1]). From now on we shall call N_i an "end" of N and we denote the total mass of N_i by M_i.

In this formulation, the (generalized) positive-mass conjecture states that for an asymptotically flat initial data set, each end has nonnegative total mass. If one of the ends has zero total mass, the initial data set can be obtained from the metric tensor and the second fundamental form of a spacelike hypersurface in the Minkowski space-time. (In particular μ and J^i must be identically zero.)

Note that York[2] has recently pointed out that the above classical formulation of the mass conjecture can be interpreted as a special case of a more general problem by relaxing the asymptotic condition on g_{ij}.

We proved the positive-mass conjecture assuming the condition that $\sum_i h_i{}^i = 0$.[3] In this Letter, we demonstrate the validity of the general conjecture by reducing it to the previous case. This will depend on a beautiful formulation of Jang[4] in his attempt to settle the positive-mass problem.

Resolution of Jang's equation.—In Ref. 4, Jang proposed to use the equation

$$\sum_{i,j} \left(h^{ij} - \frac{D^i D^j w}{(1 + |\nabla w|^2)^{1/2}} \right) \left(g_{ij} - \frac{D_i w D_j w}{| + |\nabla w|^2} \right) = 0, \quad (1)$$

where w is the unknown function with $|\nabla w| = O(1/r)$. Let

$$\bar{h}^{ij} = h^{ij} - D^i D^j w / (1 + |\nabla w|^2)^{1/2}, \quad (2)$$

$$\bar{g}_{ij} = g_{ij} + D_i w D_j w. \quad (3)$$

Then Jang observed that if one can prove that Eq. (1) is solvable, then the positive-mass problem is equivalent to proving that the total mass of the metric $\bar{g}_{ij}$ is nonnegative in each end and is zero for some end only if $\bar{g}_{ij}$ is flat and $\bar{h}_{ij} = 0$. Since Jang listed the solution of his equation as an open problem, we show here how to solve it.

We interpret the equation of Jang as follows. Consider the product of N with the real line R.

If H is the graph of w in $N\times R$ and if we extend the tensor h_{ij} to be a tensor in a trivial manner in $N\times R$, then Jang's equation says that the mean curvature of H is equal to the trace of h_{ij} with respect to the induced metric on H.

It turns out that in constructing a smooth solution to Jang's equation, we have to remove all the possible apparent horizons in N. However, the proof of the mass conjecture can still be accomplished by using a smooth solution on the complement of the apparent horizons with suitable boundary conditions. Hence for simplicity, we will assume that N contains no apparent horizons.

We solve Jang's equation first on large compact subdomains of N. To do this, we use the method of continuity. Namely, we fix a domain Ω and replace the tensor h_{ij} by the tensor th_{ij} where $0\le t\le 1$. Thus for each t, we have the equation

$$\sum_{i,j}\left(th^{ij}-\frac{D^iD^jw}{(1+|\nabla w|^2)^{1/2}}\right)\times\left(g_{ij}-\frac{D_iwD_jw}{1+|\nabla w|^2}\right)=0. \qquad (1')$$

We consider the set of t where Eq. (1′) is solvable over Ω with $w|\partial\Omega=0$. This set is nonempty because when $t=0$, the equation is trivially solvable by letting $w=0$. The set is open, from application of the implicit function theorem in a suitable form. The major part of the proof is to show that the set is closed. This depends on estimates on the solutions of Eq. (1′).

Let H be the graph of the solution of Eq. (1′) over Ω. Let e_1, e_2, e_3 be an orthonormal vector field defined locally on H and let e_4 be the normal of H. Then by translating vertically, we obtain a local orthonormal frame field in $\Omega\times R$. Let

$$\pi_{ij}=\langle D_{e_j}e_4,e_i\rangle,$$

and

$$\pi_{i4}=\langle D_{e_4}e_4,e_i\rangle,$$

for $1\le i,j\le 3$.

If we consider the tensor h_{ij} and the vector J as tensors in $\Omega\times R$, we can express them in terms of the frame e_1, e_2, e_3, and e_4. In this expression, we find

$$2(\mu-\langle J,e_4\rangle)=\overline{R}-\sum_{i,j}(\pi_{ij}-h_{ij})^2-2\sum_i(\pi_{i4}-h_{i4})^2+2\sum_i\nabla_{e_i}(\pi_{i4}-h_{i4}) +(\sum_ih_{ii})^2-(\sum_i\pi_{ii})^2+2h_{44}(\sum_ih_{ii}-\sum_i\pi_{ii})+2e_4[\sum_ih_{ii}-\sum_i\pi_{ii}], \qquad (4)$$

where $\overline{R}$ is the scalar curvature of H and Δe_i is the covariant differentiation taken with respect to the Riemannian connection of H.

By using the local energy condition, we know that the right-hand side of (4) is nonnegative. Let H' be a compact subdomain in the interior of H. Then we can multiply (4) by a suitable cutoff function; and integrating, we obtain an estimate of $\int_{H'}(\sum_{i,j}\pi_{ij}^2)$ which is independent of t. With this integral estimate of π_{ij}, we can find a pointwise estimate of π_{ij} in the interior of H' by computing $\Delta(\sum_{i,j}\pi_{ij}^2)$ and applying elliptic estimates. After providing a pointwise estimate of π_{ij} in the interior of H (which may blow up at the boundary of H) we still need an estimate of π_{ij} in a neighborhood of ∂H. To do this, we construct a barrier near ∂H to bound the gradient of w. Then we use some well-known theorems in minimal submanifold theory to bound π_{ij} in a neighborhood of ∂H.

These previous estimates are independent of t and Ω. However, if we allow the dependence on Ω, we can find estimates of all derivatives of w independent of t. The closedness of the set of t where Eq. (1′) is solvable on Ω follows readily from these estimates. In particular, we have proved that Jang's equation (1) is solvable over Ω with $w|\partial\Omega=0$.

Now we let Ω_i be a sequence of increasingly large domains in N which exhausts N. Let w_i be the corresponding solution of Jang's equation over Ω_i. Then we prove that w_i converges to a global solution w of Jang's equation with the correct asymptotic behavior in the following way.

Let H_i be the graph of w_i over Ω_i and recall that we have pointwise estimates of the second fundamental form of H_i independent of i. Using these estimates and standard elliptic theory we can prove that H_i converges to a complete, properly embedded, smooth hypersurface H without boundary in $N\times R$. The mean curvature of H is equal to the trace of P_{ij} restricted to H. It remains to be proven that H is the graph of a function w defined on N.

Let v be the vertical vector field in $N\times R$. We can estimate $\Delta\ln\langle v,e_4\rangle$ on the interior of H_i independent of i. Using this we show that $\langle v,e_4\rangle$ is bounded below by a positive constant everywhere on H or $\langle v, e_4\rangle$ is identically zero on H. In the first case H is a graph and we have shown the convergence of w_i to a limit. We show that it is impossible that $\langle v, e_4\rangle$ be identically zero on H since this would imply that H is the product of R with a two-dimensional surface in N which has, by (4), a type of stability property similar to that which we considered in Ref. 3. By arguments similar to those in Step 3 of Ref. 3 we show that the existence of such a surface is incompatible with the local energy condition unless the initial data set is trivial. Hence we have shown that $\langle v, e_4\rangle$ is positive on H and H is a graph. This shows the existence of a solution of Jang's equation (1) with the required growth properties.

Reduction of the general positive-mass problem.—Since we have demonstrated in the last section the solvability of Jang's equation, we assume the existence of smooth function w defined on N which satisfies (1) and $|\nabla w| = O(1/r)$. Since a slight modification will be enough to cover the general case, we assume that N is diffeomorphic to R^3.

The total energy of the metric $\bar{g}_{ij}$ [see (3)] is clearly equal to the total energy of g_{ij}. By using the analysis that we used in Ref. 3, we can apply the inequality (4) with $t=1$ to demonstrate the existence of a positive solution φ of the equation

$$\bar{\Delta}\varphi = \tfrac{1}{8}\bar{R}\varphi, \tag{5}$$

where $\varphi = 1 + A/r + O(1/r^2)$ with

$$A = -(1/4\pi)\int_N \tfrac{1}{8}\bar{R}\varphi[\det(\bar{g}_{ij})]^{1/2}dx. \tag{6}$$

By direct computation, the metric $\varphi^4\bar{g}_{ij}$ is asymptotically flat with total mass equal to the total mass of g_{ij} plus $\frac{1}{2}A$. Since $\varphi^4\bar{g}_{ij}$ has zero scalar curvature, the theorem that we proved in Ref. 3 shows that the total energy of g_{ij} is not less than $(1/64\pi)\int_N \bar{R}\varphi[\det(\bar{g}_{ij})]^{1/2}dx$ and equality holds only when $\varphi^4\bar{g}_{ij}$ is flat.

However, by using Eqs. (5) and (4), we can show that $\int_N \bar{R}\varphi[\det(\bar{g}_{ij})]^{1/2}dx$ is always nonnegative and is equal to zero only when $\varphi = 1$. Therefore, the total energy of g_{ij} is always nonnegative and is equal to zero only when $\varphi=1$. In the later case, the metric $\bar{g}_{ij}$ is flat by the assertion in the last paragraph. Hence $\bar{R}=0$ and we can integrate Eq. (4) to prove $\mu = \langle J, e_4\rangle$, $h_{ij} = \pi_{ij}$, and $h_{i4} = \pi_{i4}$. Since H is a graph, the local energy condition $\mu \geq |J|$ and $\mu = \langle J, e_4\rangle$ easily imply that $\mu = |J| = 0$. Similarly, we derive from the identities $h_{ij} = \pi_{ij}$ and $h_{i4} = \pi_{i4}$ the equality $\bar{h}_{ij} = 0$. This furnishes the proof of the positive-mass conjecture.

Topology of asymptotically flat space which satisfies the local energy condition.—The methods discussed previously show that over any three-dimensional manifold N which satisfies the local energy condition, there exists an asymptotically flat metric with zero scalar curvature. From this fact, we can prove that by compactifying each end of N, one can find a compact three-dimensional manifold $\bar{N}$ with positive scalar curvature so that N is diffeomorphic to $\bar{N}$ minus k points where k is the number of ends of N. Since compact three-dimensional manifolds with positive scalar curvature are basically classified in our earlier work,[5,6] we know that, modulo some standard conjectures in topology, $\tilde{N}$ is the connected sum of copies of elliptic spaces and copies of $S^2 \times S^1$. (Recall that elliptic spaces are compact three-dimensional manifolds covered by the three sphere.) One can easily verify[7] that these spaces also admit metrics with positive scalar curvature. Hence we can say that we know the topology of any asymptotically flat three-dimensional spacelike hypersurface in a space-time which verifies the energy condition.

If the space-time is the Wheeler universe, one can say the same thing about the topology because we can demonstrate the existence of a compact slice which admits a metric with positive scalar curvature.

This work was supported in part by the National Science Foundation, Grant No. MCS78-04872.

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