Stability Study of High-β Flux-Conserving Equilibria

L. A. Charlton, R. A. Dory, Y.-K. M. Peng, D. J. Strickler, S. J. Lynch, and D. K. Lee
Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830

and

R. Gruber and F. Troyon
Centre de Recherches en Physique des Plasmas, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

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The flux-conserving tokamak model suggested that rapid heating would yield equilibria with high relative energy density ($\beta = 2p/B^2$) while nonetheless allowing control over q, the so-called safety factor for instability within the ideal magnetohydrodynamic plasma model. In this study, we show that this is adequate to provide stability to β values of 10%, if there is a superconducting metal shell in the vicinity of the plasma.

High values of β (> 5%) are required for an economically credible tokamak fusion reactor. Recently, Sykes, Wesson, and Cox[1] have reported stability calculations which predicted stable plasmas with a total average β ($\overline{\beta} = 2\int p\,d\tau/\int B^2 dt, B = |\vec{B}|$) of 12% for a D-shaped plasma with an aspect ratio (A) of 2.4. Toroidal wave numbers (n) of 1, 2, and 3 were considered. Further investigation of their high-β equilibria using a model which treats high instability mode number showed that as $n \to \infty$ the critical $\overline{\beta}$ ($\overline{\beta}_c$) was only 6%, even when a stabilizing, perfectly conducting shell was assumed to be exactly at the surface of the plasma, so that kink instabilities would not have been seen.[2] Even more recent work[3] reports stability studies which found stable plasmas with β^* values of 5% [$\beta^* = 2(\int p^2 d\tau)^{1/2}/\int \beta^2 d\tau$; β^* is larger for a given plasma than β)].

We report stable equilibria with $\overline{\beta}_c = 10\%$. In our case, the perfectly-conducting-shell assumption was relaxed; we assumed a shell with a radius 20% larger than the plasma radius. A shell at this distance could represent the effect of appropriately chosen first wall material around the plasma. The shell inhibits but does not always prevent kink instabilities. Flux-conserving tokamak[4,5] equilibria with quite high β values[6] have now been analyzed with use of the computer code ERATO,[7] which treats the plasma stability within a linear ideal magnetohydrodynamic model, using a finite-element energy-prinicple approach.

Reference equilibria were generated by solving the usual equilibrium equation

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{\partial^2 \psi}{\partial Z^2} = -4\pi R^2 p' - FF'$$

with boundary conditions $\psi = \text{const}$ on a D-shaped curve given by

$$R(\theta) = R_0 + a\cos(\theta + \delta \sin\theta),$$

and

$$z(\theta) = \sigma a \sin\theta$$

with an elongation $\sigma = 1.65$, aspect ratio $A = 4.0$, minor radius $a = 1.2$ m, and $\delta = 0.5$. The plasma pressure is given by the polynomial

$$p = \beta_J[\alpha_0(x-1) + \tfrac{1}{2}\alpha_1(x^2-1) + \tfrac{1}{5}\alpha_2(x^5-1)]$$

with

$$x = \frac{\psi - \psi_{\text{axis}}}{\psi_{\text{edge}} - \psi_{\text{axis}}},$$

and $\alpha_2 = -\alpha_0 - \alpha_1$ to make $p' = 0$ at $x = 1$. An initial toroidal field function, $F = RB_T$, is specified by

$$F^2 = F_{\text{edge}} + 8\pi R_0^2[\beta_J^{-1} - 1]p.$$

In the cases presented here, an "initial" equilibrium is computed with $\beta_J = 0.5$, $F_{\text{edge}} = 2.65 \times 10^7$ kG cm, and $R_0 = 500$ cm. The values of α_0 and α_1 are adjusted so that the resulting equilibria have the specified values of q_0 (safety factor at the magnetic axis) and q_s (safety factor at the surface). The higher values of β_p ($= 2\int p\,d\tau/\int B_p^2 d\tau$, where B_p is the poloidal component of $\vec{B}$) are then obtained by increasing the parameter β_J and recomputing the equilibria, using a numerical method which conserves $q(\psi)$ and $\psi_{\text{edge}} - \psi_{\text{axis}}$. The reference equilibria were then continuously scaled[8] to reduce $\overline{\beta}$ until stability was found at a critical value $\overline{\beta}_c$. The values of β_p and $q_s q_0$ are approximately constant during the scaling, and so they can be used as labels for the equilibria. We studied nine equilibria with $\beta_p = 1$, 2,5, and 3.5, and $q_s/q_0 = 2$, 3, 4.

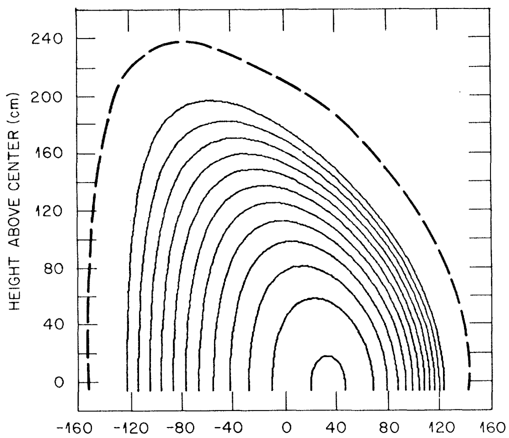

FIG. 1. Flux contours for the equilibria with the highest $\overline{\beta}_c$ (=10%). The dashed line gives the location of the conducting shell.

Figure 1 shows the flux contours for the equilibria with $\beta_p = 2.5$ and $q_s/q_0 = 2.0$, which gave the highest $\overline{\beta}_c$. A conducting shell was assumed to lie at the location of the dashed line. The pressure profiles for these equilibria were bell-shaped as expected, while the current density profile was peaked toward the outside, as is typical of high-β flux-conserving tokamak equilibria.[6] The q profiles increased monotonically from the axis to the edge. They had a very flat character but were otherwise unremarkable.

Figure 2 shows regions of stability for the $\eta = 1$ mode as a function of q_s. The curves define constant-β_p contours with values 1.0, 2.5, and 3.5. On each curve, q_s/q_0 varies from 4 at the bottom to 2 at the top. Smaller q values correspond to higher $\overline{\beta}_c$. Diagrams for $n = 2$, 3, and 4 are similar. An optimum β_p is observed to occur near a value $\beta_p \sim A/2$. The value of q_0 is approximately the same along the constant-β_p lines with larger

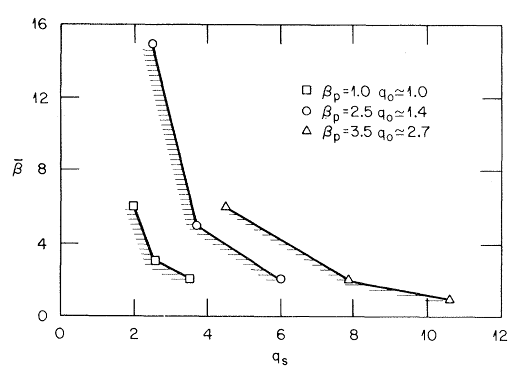

FIG. 2. $\overline{\beta}$ vs q_s for $n = 1$. The shaded region to the left of each curve is stable.

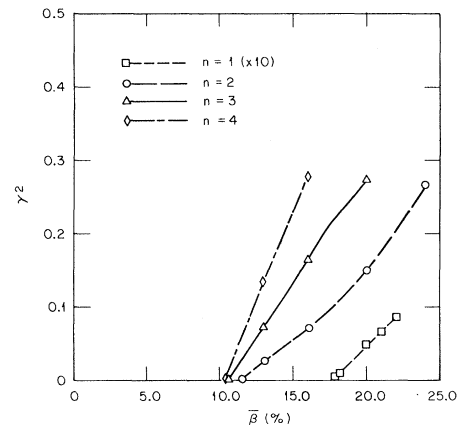

FIG. 3. γ^2 vs $\overline{\beta}$ for $n = 1$, 2, 3, and 4 using the equilibria with the highest $\overline{\beta}_c$.

values at higher β_p.

The highest value found for $\overline{\beta}_c$ occurs at $q_s/q_0 = 2.0$ and $\beta_p = 2.5$ for $n = 1$, 2, 3, and 4. Figure 3 shows the β dependence of the instability growth rate. There is a region of stability for $n = 1$ at very high β (> 30%); this was not studied further because higher-n modes were very unstable for the cases treated and in the high-β region $q_0 < 1$ which would lead us to expect resistive instabilities[9] which are not included in the present analysis. Tearing modes are found experimentally to limit discharges to $q_0 \gtrsim 1$.

All the results shown are based on convergence studies and extrapolation to infinitesimal numerical grid spacing. Figure 4 shows the growth rates obtained using 30, 35, 40, and 45 grid points in both the radial and poloidal directions. A poly-

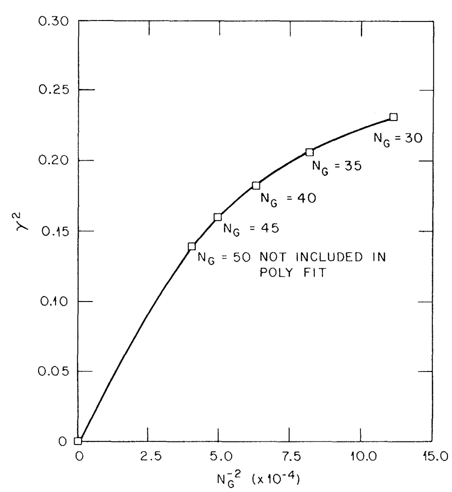

FIG. 4. The square of the growth rate vs N_G^{-2} for $n = 4$ (N_G is the number of grid points). A polynomial fit was used to find the growth rate as $N_G \to \infty$.

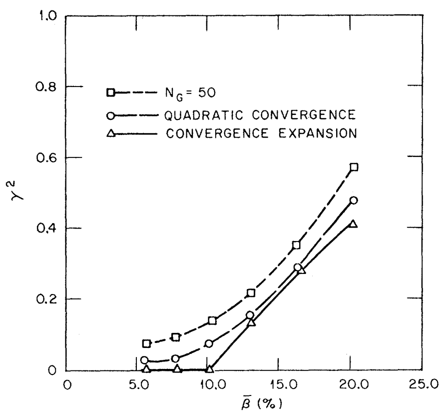

FIG. 5. The square of the growth rate vs $\bar{\beta}$ using (1) 50 grid points ($N_G = 50$), (2) convergence study assuming N_G^{-2} convergence, and (3) convergence study using a polynomial fit (convergence expansion).

nomial expansion in the grid spacing was used to fit the curve; the 50-grid-point data served as a check on the fitting procedure, and the growth rates were then extrapolated to zero grid spacing values. Figure 5 shows the dependence of the squared growth rate on $\bar{\beta}_c$ when $N_G = 50$ grid points is used, when quadratic convergence is assumed ($\gamma^2 = a + b/N_G^2$, where N_G is the number of grid points), and when a quartic convergence expansion is used. As is seen, it is essential that a complete convergence study be used.

Figure 6 shows the dependence of $\bar{\beta}_c$ on toroidal wave number. The point near $n = \infty$ was obtained from ballooning theory.[10, 11] Direct evaluation at $n = \infty$ gives instability but inclusion of $1/n$ corrections[11] suggests that the unstable region lies only at very high n values where validity of the magnetohydrodynamic theory is suspect because of finite-gyroradius and kinetic effects. Applying the correction

$$\gamma^2 = \gamma_0^2 + \frac{1}{2n\,|\nu'(\chi_0)|}\left(\frac{\partial^2\gamma^2}{\partial\chi_0^2}\,\frac{\partial^2\gamma^2}{\partial\psi^2}\right)^{1/2}, \tag{1}$$

where γ_0 is the $n = \infty$ growth rate, ν' is the local shear, and χ_0 is the lower limit of integration in computing the global shear (see Ref. 11), we find a critical n of 150. This large $1/n$ correction (which cancels the first-order term) occurs because ν' is nearly zero at the most unstable ψ surface. We have applied corrections only to order $1/n$, and these are sufficiently large to cause concern over the convergence of this expansion for the cases treated. As a result, this usual expansion probably provides only a rough estimate of the finite-n corrections. This point is under further investigation.

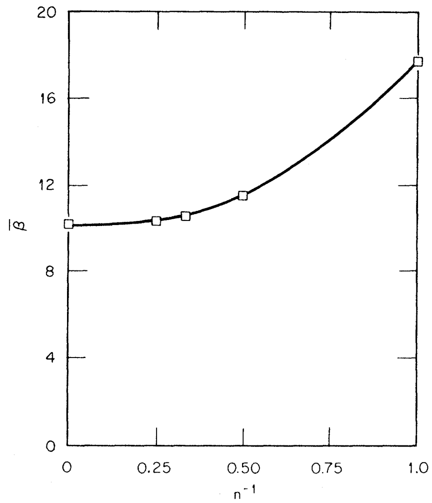

FIG. 6. $\bar{\beta}_c$ vs n^{-1}.

The results for all the equilibria and all the toroidal wave numbers have roughly the same quantitative behavior as shown in Fig. 2 for $n = 1$, except for a progressively lower $\bar{\beta}_c$ as n is increased. Thus, we can use the data to suggest a scaling law—at least for the regime in the present study. The results in Fig. 2 suggest a relation

$$\bar{\beta}_c = U(\beta_p/A)^{3/2}/q_s^2A, \tag{2}$$

where U (for $n \to \infty$) is ~8.6 in this study, but must be regarded as a function of all plasma parameters not varied in the present results. The functional form in Eq. (2), however, does not include the optimum β_p behavior exhibited in Fig. 2. The rough constancy of q_0^c (the critical safety factor on axis) along each line in Fig. 2 suggests

$$q_0^{c\,2} = (1 - \beta_p/A)^{-1}. \tag{3}$$

Shafranov and Yurchenko[12] found similar q_0^c dependence from a Mercier criterion (localized mode) study. The inclusion of Eq. (3) gives

$$\bar{\beta}_c = \{Uq_0^2(\beta_p/A)^{3/2}(1 - \beta_p/A)\}/q_s^2A. \tag{4}$$

Equation (4) is an empirical representation of a large set of results for equilibria with $A = 4.0$. Additional equilibria with different aspect ratios are now being studied to ascertain the A dependence shown in Eq. (4).

It is our conclusion that stability requirements for a D-shaped plasma with elongation of 1.65 allow stable average β's as high as 10%, even at a high aspect ratio of 4, with $q_s/q_0 = 2.0$ and $\beta_p = \frac{1}{2}A$. Further studies of the dependence of U on the plasma shape, the elongation, and the proximity of the conducting shell may lead to higher β values, while future analysis of resistive and kinetic instabilities may reduce performance somewhat. In any event, the present results lead us to be significantly more encouraged than was the case earlier.

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Tearing Modes in a Plasma with Magnetic Braiding

P. K. Kaw, E. J. Valeo, and P. H. Rutherford
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544
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This paper describes examination of certain linear and nonlinear properties of macroscopic tearing modes driven by anomalous electron viscosity effects associated with magnetic braiding. It is shown that strong linear growth of $m=1$, 2 tearing modes (respectively proportional to $\mu^{1/5}$ and $\mu^{1/3}$) can occur for rather modest values of μ. In the nonlinear phase, the island width grows in time as $t^{1/3}$. Some speculations regarding the disruptive instability in tokamaks are also investigated.

In recent years, the idea that a tokamak may suffer from broken magnetic surfaces and braided magnetic field lines has attracted a lot of attention.[1-3] It is believed that magnetic braiding could result if a number of helical magnetic perturbations centered on different mode-rational surfaces are simultaneously excited in the plasma. The possible existence of braided field lines has variously been considered as an important candidate for explaining certain observed phenomena in tokamaks, such as the anomalous electron heat transport,[1-3] aspects of disruptive instability,[1-4] behavior of runaway electrons,[5] etc.

If field-line braiding arises because of quasistatic, small-scale magnetic perturbations, it is likely that it will lead not only to an anomalous transport of electron heat[1,2] but also to an equally strong transport of electron parallel momentum across unperturbed magnetic surfaces.[1] This means that Ohm's law must be modified to incorporate an anomalous electron viscosity coefficient, which leads to a smoothing out of perpendicular gradients of parallel current, i.e.,

$$E_{\parallel} = \eta J_{\parallel} - \bar{\mu}(m/ne^2)\nabla_{\perp}^2 J_{\parallel}; \tag{1}$$

here η is the resistivity and the viscosity $\bar{\mu}$ is approximately equal to χ, the coefficient of anomalous perpendicular electron thermal conductivity in a braided field.[1,2] This result is consistent with a simple quasilinear estimate of the influence of microscopic magnetic perturbations on