ture [see P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 1294]. For a ring angular velocity, $\dot{\theta}$, the circulation contribution arising from the orifice is $-\dot{V}/2a$ where $\dot{V}=(1-\chi)AR\dot{\theta}$ is the volume rate of flow through the orifice. Here we neglect the velocity of the partition, $R\dot{\theta}$, as compared to the mean flow velocity, $\bar{v}==V/\pi a^2$, through the orifice. The contribution to the circulation from flow in the rest of the ring is $\chi 2\pi R^2\dot{\theta}$. The requirement of zero total circulation leads to the given expression.

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# Rotational Speedups Accompanying Angular Deceleration of a Superfluid

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Exact calculations of the angular deceleration of superfluid vortex arrays show momentary speedups in the angular velocity caused by coherent, multiple vortex loss at the boundary. The existence and shape of the speedups depend on the vortex friction, the deceleration rate, and the pattern symmetry. The phenomenon resembles, in several ways, that observed in pulsars.

The angular deceleration ("spin-down") of a superfluid has gained much interest from the identification of pulsars as neutron stars and the well-founded prediction that the latter consist primarily of a neutron superfluid.[1,2] In particular, the abrupt changes in the rotation period, variously known as jumps, speedups, or anomalies ("glitches"), possibly originate in superfluid hydrodynamic processes.[3-6] Continuous vorticity models have not provided a mechanism for rotational anomalies although various vibration[7,8] and relaxation[9] phenomena have been studied. The new results of this Letter concern basic dynamical characteristics of discrete vortex arrays during unconstrained spin-down. These characteristics obviously invite comparison with those of pulsars, although only a relatively small number of vortices are used and only the grossest pulsar features are mirrored in the theoretical system.

I consider a rotating, two-component superfluid in which the vortices have singly-quantized circulation $\kappa$ and are conserved except for possible annihilation at an exterior, circular boundary. To be consistent with the pulsar system, I assume the effective normal fluid is locked rigidly to the boundary (because of the strong magnetic field in neutron stars[1,2]). Also, the vortices are assumed to be unpinned at the boundary (a result of either the loss of $^3P_2$ superfluidity there or the existence of a boundary layer).[5] Thermodynamic processes and any difference between vortex dynamics in a sphere and a cylinder are disregarded for convenience.

The friction accompanying relative motion of the vortices and the normal fluid can be expressed as an angle $\theta$ relating the velocities of the $k$th vortex, $v_k$, of the local superfluid, $v_{s,k}$, and of the local normal fluid, $v_{n,k}$. In complex notation, appropriate for rectilinear vortices, this expression is[10]

$$v_k=e^{-i\theta}\left[v_{s,k}\cos\theta+iv_{n,k}\sin\theta+\frac{\nu'}{\rho_s\kappa-\nu'}(v_{s,k}-v_{n,k})\cos\theta\right], \qquad (1)$$

 

where $\nu'$ is the transverse component of friction which, for convenience, is taken to be zero. In the normal-fluid frame, $\theta$ is the angle between the vortex and superfluid velocities; its physical extremes are 0° (no friction, vortices move with superfluid) and 90° (maximum friction, vortices locked to normal fluid).

In a nonrotating frame, the superfluid velocity $v_{s,k}$ at the vortex position $z_k$ is due to the other vortices and all the images,[11] so that

$$v_{s,k} = i\sum_{j\neq k}^{N}\frac{1}{\bar{z}_k - \bar{z}_j} - i\sum_{j=1}^{N}\frac{1}{\bar{z}_k - 1/z_j}, \qquad (2)$$

and the normal-fluid velocity, for solid-body rotation, is just $v_{n,k} = i\omega z_k$. I use dimensionless angular velocity $\omega = 2\pi R^2\Omega/\kappa$, and time $\tau = \kappa t/2\pi R^2$, and normalize $z_k$ to the boundary radius $R$.

The calculations begin with an initial vortex pattern chosen from the Los Alamos Catalog (LAC).[12] With use of the above equations, the pattern is first put in equilibrium at a starting $\omega$ and then, after assigning inertia to the boundary and normal fluid, spun down according to some fixed prescription for the external decelerating torque. The torque acts on the boundary, which is coupled to the other components as described above. The resulting decrease in $\omega$, a dependent variable, causes the pattern to grow beyond its stability size and lose vortices at the boundary.

The dynamics of the system shows distinctive behavior. For larger average decelerations, $\dot{\omega} \equiv \langle d\omega/d\tau\rangle$, and larger vortex friction angles $\theta$, groups of vortices are lost simultaneously at comparable intervals of $\omega$. At smaller values of $\dot{\omega}$ or $\theta$, it is common for the vortices to leave individually in an irregular manner but at an average rate proportional to $\dot{\omega}$; this will be called incoherent loss (IL). Such IL, if any, typically sets in after one or more instances of the coherent vortex loss, which results in a pronounced speedup event in the angular velocity as shown in Fig. 1. The number of vortices lost per event also depends on the symmetry of the initial pattern, with those of higher symmetry losing more. The influence of the initial pattern extends through most of the decay. For example, the intermediate patterns that occur during the spin-down of LAC $61_1$ are quite different from those that occur from LAC $50_3$. From the start, unsymmetric initial patterns lose vortices by the IL mode; for example, at $\dot{\omega} = -1$, pattern LAC $37_1$ loses only coherent groups but LAC $37_3$ shows only IL. The angular momemtum of the superfluid abruptly drops when vortices are lost and then recovers to an intermediate level as shown in Fig. 2. This recovery is caused by both the consolidation of the new pattern and its contraction in response to the transiently higher value of $\omega$ at the peak of the speedup. The contraction increases the vortex angular momentum and hence must decrease $\omega$, to conserve the total angular momentum. The

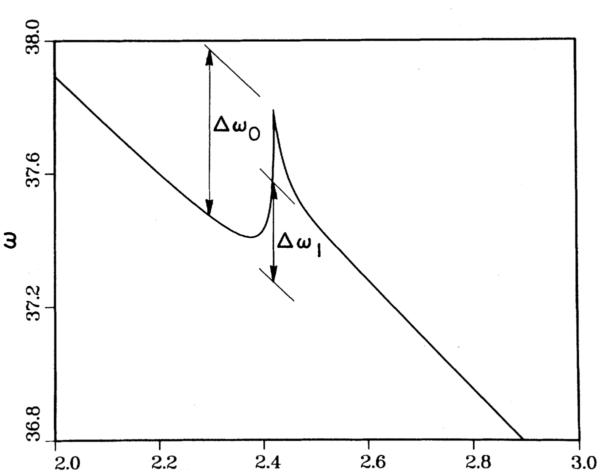


FIG. 1. Angular velocity vs time (dimensionless) for the first speedup event during the deceleration of LAC $37_1$, with $\dot{\omega} = -1.5$, $\theta = 45°$. Six vortices are lost. The $Q = 1 - \Delta\omega_1/\Delta\omega_0 = 0.41$.

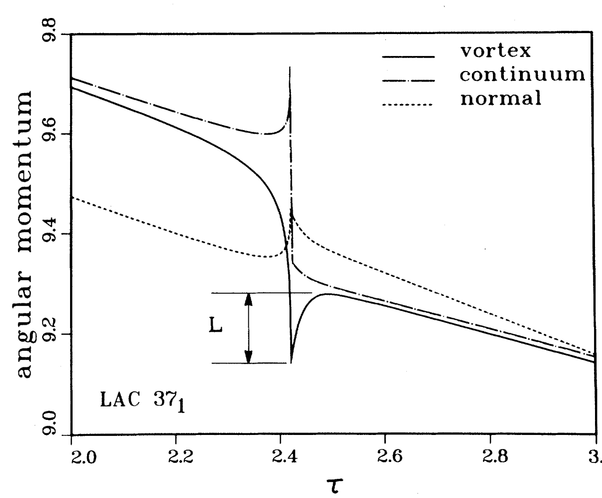


FIG. 2. Angular momentum, in units of $\frac{1}{2}\rho\kappa R^2$, for the event in Fig. 1. The continuum approximation, $(\rho_s/\rho)N[1-(N-1)/2\omega]$, of the vortex angular momentum fails completely at the jump. Here, $\rho_s/\rho = 0.5$. Not shown is the angular momentum of the boundary, which is, arbitrarily, twice that of the normal component.

vortex energy has a similar recovery but, unlike the angular momentum, most of its net change is impulsively dissipated rather than reversibly transferred to the nonsuperfluid components.

It is interesting to compare the above with pulsar jump phenomena:

(1) Both the vortex and pulsar speedups are always positive with a fast rise time $\tau_1$ followed by a slower decay $\tau_2$ to an offset and a longer interval $\tau_3$ between events, i.e., $\tau_1 \ll \tau_2 \ll \tau_3$. In the vortex system, $\tau_1$ is characteristic of vortex motion dominated by image interactions, $\tau_2$ is the pattern equilibration time (recovery), and $\tau_3$ is proportional to $\dot{\omega}^{-1}$.

(2) Although hundreds of pulsars have been discovered, jumps have been seen mostly in the youngest, those with the highest $\omega$, largest deceleration $\dot{\omega}$, and perhaps, the highest temperature (and vortex friction).[13] As noted above, it is important for $\dot{\omega}$ and $\theta$ to be sufficiently large if coherent rather than incoherent loss is to occur in the vortex system. The analog of IL in pulsars could not cause speedups because the unit change in vortex angular momentum is too small. (However, it could produce noise in the timing of the pulsar signals.) Figure 3 shows a spin-down of LAC $61_1$ with torque proportional to $\omega^2$. After six pronounced speedups, the IL mode begins when $\dot{\omega}$ becomes sufficiently small. The IL mode could also have been induced by decreasing $\theta$, even with constant torque.

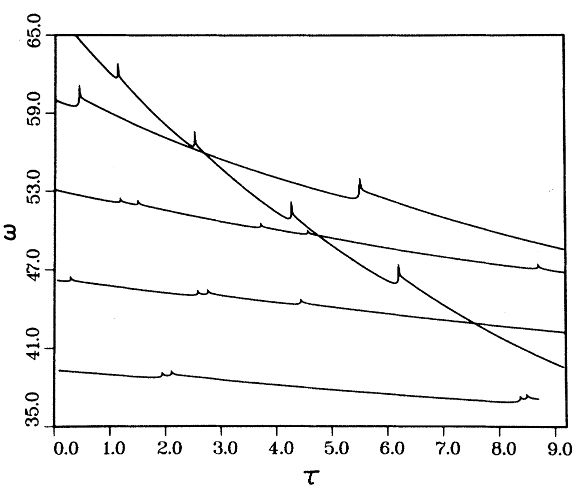


FIG. 3. Deceleration of LAC $61_1$ with decreasing torque. After six events (losing six vortices each) the incoherent decay begins (irregular loss of individual vortices). The function $\omega(\tau)$ for $\tau > 9.2$ is successively returned to the origin and displaced upward but retains its scale. $\theta = 30°$.

(3) A most puzzling feature of pulsar period jumps has been the difference in the shape parameter $Q \equiv 1 - \Delta\omega_1/\Delta\omega_0$ between the Crab and Vela events (see Fig. 1). The puzzle arises from relating $Q$ to the system's properties. If the system consists of loosely coupled rigid and liquid components having angular momenta $I_c$ and $I_l$, and if the jump is caused by a small change in $I_c$, then, by a simple argument based on conservation of angular momentum,[14] $Q \approx I_l/(I_c + I_l)$. The fact[5] that typical $Q_{\text{crab}} \approx 0.9$ and $Q_{\text{Vela}} \approx 0.15$ implies, by this relation, a strange and enormous difference in structure between the two pulsars which are not otherwise expected to be so different. For the vortex system, the puzzle disappears because the above relation for $Q$ is not applicable. The vortex angular momentum is not proportional to $\omega$ and the source of the speedup is a vortex instability. In this case, conservation of angular momentum gives the estimate

$$Q \approx [(1-\epsilon)I_n + L/\Delta\omega_0]/(I_n + I_c), \tag{3}$$

where $I_n$ is the moment of inertia of the normal fluid, $L$ is the amount of vortex angular momentum change during the recovery period $\tau_2$, and $\epsilon$ is the fraction of normal fluid effectively locked to the crust by viscosity during $\tau_1$. (Elsewhere it is assumed $\epsilon = 1$.) The effective moment of inertia of the superfluid, the dominant star component, does not appear in Eq. (3). As a result, vortex systems of *identical* size and composition can exhibit speedups with widely different $Q$ values due to the variability of $L/\Delta\omega_0$. In particular, smaller $\dot{\omega}$ gives smaller $Q$ as shown qualitatively in Fig. 3 where the $Q$'s steadily decrease until the IL mode begins. Also, reducing the vortex friction angle $\theta$ results in lower $Q$ as shown in Fig. 4. To some extent this is due to longer relaxation times at smaller $\theta$ so that in a fixed time period the $Q$ appears smaller. Both of these vortex mechanisms are consistent with the pulsar observations: $\dot{\omega}_{\text{Crab}} > \dot{\omega}_{\text{Vela}}$ and the vortex friction in the younger and, presumably, hotter Crab pulsar is, plausibly, greater than in the Vela.

Because of the disparity between the number of vortices used in the calculations ($\leq 61$) and the number in the Crab or Vela pulsar ($\approx 10^{17}$), the calculated speedups represent the loss of only a fraction ($\approx \frac{1}{2}$) of an outer ring compared with the equivalent loss of many rings ($\approx 10^4$) in these pulsars, assuming their vortices are lost only during the observed jumps. Vortex dynamics, if it is to explain pulsar jumps, must either

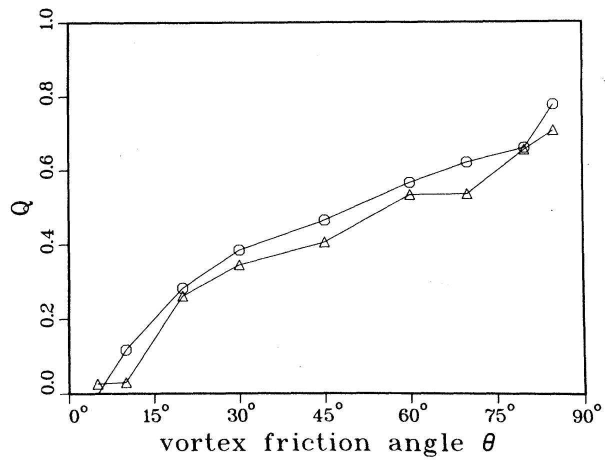


FIG. 4. Dependence of the shape parameter $Q$ on the vortex friction angle $\theta$ for the first event LAC $37_1$. Triangles: $\dot{\omega}=-1.5$. Circles: $\dot{\omega}=-2$. For $\theta \leq 30°$ incoherent loss begins to occur in the later stages of the deceleration (at constant torque).

cause the abrupt loss of many rings or trigger other mechanisms that amplify the consequences of smaller vortex loss. This may require adding to the dynamics new elements such as thermal processes,[15] the variation of vortex friction over space and time, and pinned vorticity ($^1S_0$ superfluid) within the crust.[16] However, with respect to the total vortex number the relative fractions lost are reversed, of order $\frac{1}{6}$ for the events shown here and $10^{-4}$ for the pulsars. It is favorable that the larger speedups came from patterns having pronounced triangular symmetry, the dominating symmetry of much larger patterns.

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*Note added.*— L. V. Kiknadze and Yu. G. Mamaladze, Zh. Eksp. Teor. Fiz. 75, 607 (1978) [Sov. Phys. JETP 48, 305 (1978)] consider some of the phenomena discussed here and show results from the deceleration of LAC $19_1$. I am grateful to Dr. Mamaladze for bringing this to my attention.

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