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Bose-Einstein Condensation of a Relativistic Gas in d Dimensions

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This Letter investigates the conditions for Bose-Einstein condensation of an ideal relativistic Bose gas in different spatial dimensions. It is shown that the thermodynamical properties of the critical particle density are qualitatively different for massive and massless relativistic Bose particles.

Our aim in this work is to show that a relativistic Bose gas leads to a different dimensional dependence of the Bose-Einstein condensation (BEC) in the cases of massive and massless Bosons. The effect of dimensionality has been previously investigated¹⁻³ for nonrelativistic thermodynamical systems possessing an energy spectrum of the form

$$E = \sum_{i=1}^d c_i p_i^\alpha, \quad (1)$$

where c_i are given volume-dependent factors, d is the number of spatial dimensions, p_i are the components of the momentum in the direction i , and α is a free parameter characterizing an external field. The case $\alpha=2$ corresponds to the nonrelativistic ideal gas, whereas $\alpha=1$ describes a system which is composed of harmonic oscillators. It was also shown¹ that the relevant quantity which determines whether or not the system can condense is given by d/α ; only for $d/\alpha > 1$ can the BEC take place. Furthermore, the critical properties of the nonrelativistic BEC as a

cooperative phase transition have been exactly calculated⁴ for arbitrary d ; the importance of the continuous values of d is related to the parameter $\epsilon = 4 - d$ in renormalization-group calculations.^{5,6}

Now we shall consider a system of relativistic bosons, each with a rest mass m and the relativistic energy spectrum ($c = \hbar = 1$)

$$E = (m^2 + |\vec{p}|^2)^{1/2} \quad (2)$$

contained in a d -dimensional spatial volume V_d . The d -dimensional invariant phase-space measure⁷ $d\sigma_d(0)$ is written as

$$\begin{aligned} d\sigma_d(p) &= 2 \frac{V_d^{(d)} p^\mu}{(2\pi)^d} \theta(p_0) \delta(p^2 - m^2) d^{d+1}p \\ &= \frac{1}{(2\pi)^d} \frac{V_d^{(d)} p^\mu}{p_0} d^d p \end{aligned} \quad (3)$$

which is readily seen to be the generalization of the relativistic ideal gas⁸ in the usual covariant formulation⁹ for three spatial dimensions. From this expression we may obtain the thermodynamical

cal potential Ω as the logarithm of the grand partition function¹⁰ in the form

$$-\beta\Omega = \frac{V_d}{(2\pi)^d} \int d^d p \ln(1 - Ae^{-\beta E}) - \ln(1 - Ae^{-m\beta}), \quad (4)$$

where β is the inverse temperature and A is the relativistic fugacity⁹ with a direct relation to the chemical potential μ by $A = \exp(\beta\mu)$ and to the usual nonrelativistic fugacity z through $A = z \exp(\beta m)$. The presence of zero-momentum states leads to the term $\ln(1 - Ae^{-m\beta})$; its contribution becomes

important as A approaches $\exp(\beta m)$. This term gives rise to the well-known phenomenon of BEC for a nonrelativistic Bose gas in d dimensions with $z \rightarrow 1$. For the usual nonrelativistic system it is furthermore known that for all dimensions $d > 2$ and at all finite densities below a critical temperature the zero-momentum state contains a certain finite fraction of the particles.

The aim of our present investigation of the BEC is basically twofold: (1) the numerical study of lower dimensions and the determination of the conditions for which the condensation disappears and (2) the determination of limit of large dimensions. We assume a spherically symmetrical system so that Eq. (4) becomes

$$-\beta\Omega = \frac{V_d}{(2\pi)^{d-1}} \frac{\pi^{(d-2)/2}}{\Gamma(d/2)} \int_0^\infty dp p^{d-1} \ln(1 - Ae^{-\beta E}) - \ln(1 - Ae^{-m\beta}). \quad (5)$$

The evaluation of this integral is carried out through an expansion with use of the integral representation for the modified Bessel functions of the second kind¹¹ in the form

$$K_\nu(x) = \frac{\Gamma(1/2)}{\Gamma(\nu+1/2)} \left(\frac{x}{2}\right)^\nu \int_1^\infty e^{-xt} (t^2 - 1)^\nu dt, \quad (6)$$

where x , t , and ν are positive real numbers and Γ is the gamma function. Eq. (5) then takes the form

$$\Omega = - \frac{V_d \pi^{(d-3)/2} (2m/\beta)^{(d+1)/2}}{2(2\pi)^{d-1}} \sum_{k=1}^\infty \frac{A^k}{k^{(d+1)/2}} K_{(d+1)/2}(km\beta) + \frac{1}{\beta} \ln(1 - Ae^{-m\beta}). \quad (7)$$

We obtain the average particle number density from the usual relation

$$n = -(A/V_d) \partial(\beta\Omega)/\partial A \quad (8)$$

which with Eq. (7) gives

$$n = \frac{\pi^{(d-3)/2}}{2(2\pi)^{d-1}} \frac{(2m)^{(d+1)/2}}{\beta^{(d-1)/2}} \sum_{k=1}^\infty \frac{A^k}{k^{(d-1)/2}} K_{(d+1)/2}(km\beta) + \frac{A}{V_d} \frac{e^{-m\beta}}{1 - Ae^{-m\beta}}, \quad (9)$$

where the second term corresponds to the zero-momentum states as in the usual BEC with $d=3$. The structure of the expansion (9) shows that for the relativistic gas¹² it is useful to introduce a dimensionally dependent quantity

$$L_d(m, \beta) = \frac{\pi^{(d-3)/2}}{\beta^d} \frac{2^{(d-1)/2}}{(2\pi)^{d-1}} (m\beta)^{(d+1)/2} e^{m\beta} K_{(d+1)/2}(m\beta), \quad (10)$$

which yields the proper relationship between the density and fugacity for the ideal relativistic Boltzmann gas, $n = L_d z$. For the Bose gas we find that

$$n = L_d \sum_{k=1}^\infty \frac{z^k e^{(k-1)m\beta}}{k^{(d-1)/2}} \frac{K_{(d+1)/2}(km\beta)}{K_{(d+1)/2}(m\beta)} + \frac{1}{V_d} \frac{z}{1-z}. \quad (11)$$

In the ultrarelativistic limit ($m\beta \rightarrow 0$), L_d leads to $2\Gamma(d)/\Gamma(d/2)(1/2\pi)^{d/2}\beta^d$, which for $d=3$ reduces to twice the optical wavelength¹³ to the third power and in the nonrelativistic limit ($m\beta \rightarrow \infty$) it leads to λ^{-d} , where $\lambda = (2\pi\beta/m)^{1/2}$ is the thermal wavelength.¹⁰

In order to examine the conditions for the BEC more closely, we rewrite Eq. (11) in such a way that only the condensation term $L_d^{-1} z/V_d(1-z)$

remains on the right-hand side. Analogous to the usual¹⁰ BEC the nonnegativity of this term for $0 \leq z < 1$ provides an inequality which in the limits $V \rightarrow \infty$ and $z \rightarrow 1$ yield

$$L_d^{-1} n \geq \sum_{k=1}^\infty \frac{\exp[(k-1)m\beta]}{k^{(d-1)/2}} \frac{K_{(d+1)/2}(km\beta)}{K_{(d+1)/2}(m\beta)}, \quad (12)$$

where the equality defines a critical density n_c .

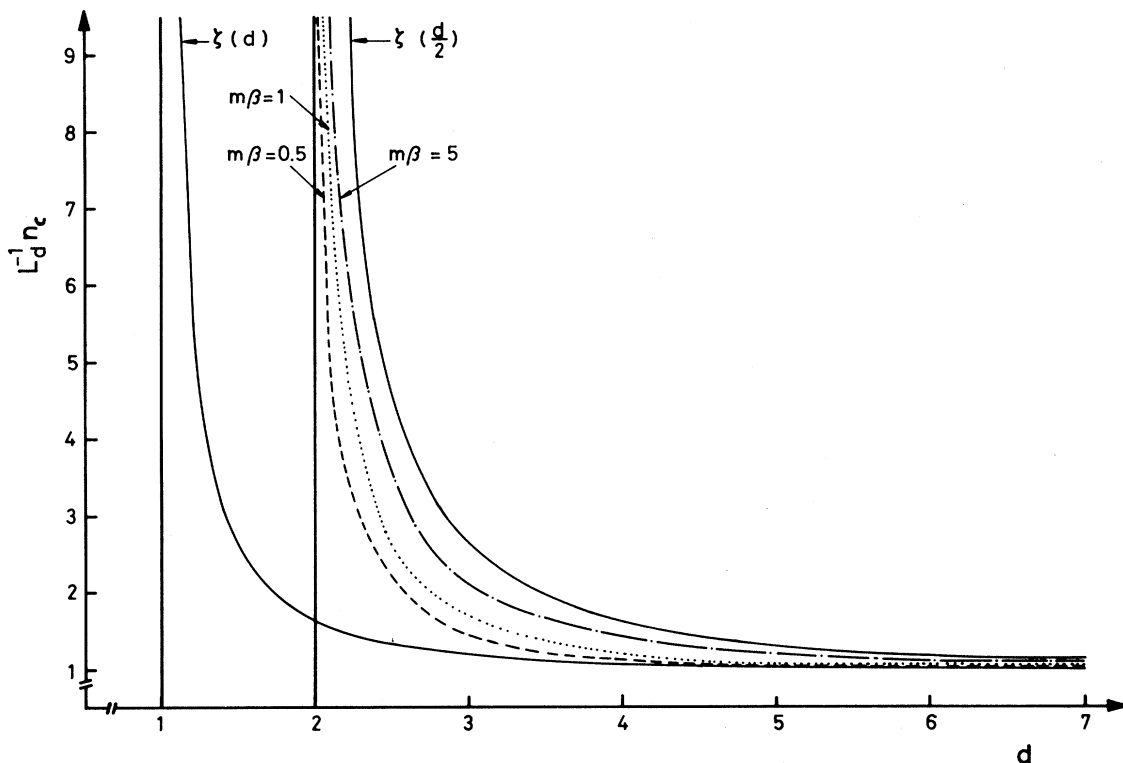


FIG. 1. The ratio of the critical density of a Bose gas to the corresponding Boltzmann gas as a function of the spatial dimensions for various values of $m\beta$.

One can see that $L_d^{-1}n_c$ is just the critical density of a Bose gas divided by the corresponding density of a Boltzmann gas of the same fugacity.

The meaning of this criterion for BEC is well understood¹³ for $d=3$. With a bit of algebraic manipulation, together with the use of the asymptotic forms¹¹ for $K_{(d+1)/2}(x)$ in the limit $x \rightarrow \infty$, we are able to find for $m \neq 0$ the upper bound of the sum in Eq. (12). It involves the Riemann zeta function¹⁴ $\zeta(d/2)$ which arises directly in the expression $n\lambda^3$ for the nonrelativistic Bose gas.¹⁰ Likewise in the limit $x \rightarrow 0$ and $m \neq 0$, we can show that $\zeta(d)$ serves as a lower bound for Eq. (12). For $m=0$, we have simply $\zeta(d)$ for the sum in Eq. (12). The result of these considerations is shown in Fig. 1 where we give the numerical evaluation for certain cases. When $d > 2$ the values of $L_d^{-1}n_c$ remain finite for all particles of finite mass m ; BEC thus takes place for dimensions $d > 2$. Only the case with $m=0$ (photon gas) has a finite value at $d=2$, as listed in Table I; however, it diverges at $d=1$, because of the divergence of $\zeta(1)$. In Table I we have also listed the results of our calculations for the subsequent integer dimensions. The values for $\zeta(z)$ have been taken from Dwight.¹⁵ It should be noted that

for large dimensions $L_d^{-1}n_c$ of the relativistic Bose gas is such that $L_d^{-1}n_c$ approaches 1. Since $\zeta(d) \leq L_d^{-1}n_c \leq \zeta(d/2)$ serve as lower and upper bounds while going to 1 as d is taken to infinity. This relationship implies that the limit $d \rightarrow \infty$ provides the classical Boltzmann gas. From this fact it appears that the large number of dimensions leads to a condensationlike effect for classical systems.

Our calculations of the critical density for the occurrence of BEC have shown that there is a qualitative difference between the behavior of massive and massless Bose gases, in that all massive cases have condensation only for $d > 2$. If we choose an energy spectrum (2) in the form $(m^2 + |p|^\alpha)^{1/2}$, we may readily generalize our results. By going through our arguments again we also find that the condensation is present whenever $d/\alpha > 1$. The massless Bose gases, however, always have an energy spectrum which depends linearly on the momentum coordinates. Such systems are, therefore, closely related to the quantized harmonic oscillator.¹ The qualitative difference between $m=0$ and $m \neq 0$ clearly shows that, in the vicinity of the transition region, massive Bose gases in lower dimensions

TABLE I. The ratio of the critical density of a Bose gas to the corresponding Boltzmann gas in various dimensions and for different values of $m\beta$. The nonrelativistic [$\zeta(d/2)$] and ultrarelativistic [$\zeta(d)$] limits are included.

d	$\zeta(d/2)$	$m\beta = 10$	$m\beta = 5$	$m\beta = 1$	$m\beta = 0.5$	$m\beta = 0.1$	$\zeta(d)$
2	∞	∞	∞	∞	∞	∞	1.645
3	2.612	2.406	2.084	1.665	1.460	1.251	1.202
4	1.645	1.522	1.451	1.241	1.150	1.095	1.082
5	1.341	1.245	1.214	1.089	1.062	1.043	1.037
6	1.202	1.135	1.112	1.041	1.028	1.021	1.017
7	1.127	1.100	1.061	1.020	1.013	1.012	1.008

cannot be well approximated by the corresponding ultrarelativistic (massless) case. This distinction has not always been clearly stressed.¹⁶ Furthermore, we want to point out that our results on the occurrence of BEC can also be seen from the phase-space integral for the average particle number of the form $\int_0^\infty dp p^{d-1} \{e^{\beta[E(p)-\mu]} - 1\}^{-1}$. The upper limit of integration provides no problem because of the dominant exponential structure. However, the lower limit of small momenta reduces to an integrand of the form $p^{d-1}/[E(p) - \mu]$, which by expansion of Eq. (2) yields p^{d-3} . Thus this integral diverges at and below $d = 2$. The ultrarelativistic case from a similar argument has an integrand of the form p^{d-2} , which first diverges at $d = 1$. The integral $\int dp/p$ in general possesses a simple logarithmic divergence as seen in Fig. 1. The absence of a dimensional scale provides the physical basis for the termination of the BEC.

The observed qualitative difference between the massless and the massive Bose gas at the condensation point shows itself most drastically through the anomalous behavior of the specific heat c_v in three dimensions.¹⁷ We find that the massless Bose gas exhibits a discontinuity Δc_v for all $d > 2$ while its massive counterpart yields this structure first for $d > 4$, so that

$$\Delta c_v(d/2, m=0) = \lim_{m \rightarrow \infty} \Delta c_v(d, m \neq 0). \quad (13)$$

This result¹⁸ corresponds directly to the work which we have presented here for the particle density as the determiner of BEC from the ratio d/α .

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