Bifurcation-Theory Approach to Electron Solids in Superstrong Magnetic Fields

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Bifurcation theory is used to study the transition from a low-density nondegenerate electron gas to an "electron solid," in a superstrong magnetic field. The dynamics of the gas are described by a set of nonlinear equations, and the transition is seen as the bifurcation of spatially periodic solutions from the trivial constant solution. Electron solids are a good approximation for condensed matter in the outer crust of magnetic neutron stars.

Condensed matter in the outer crust of pulsars has peculiar properties due to the enhancement of the electron motion by magnetic fields of the order of 10^{12} G. Various models have described this structure in terms of bundles of "magnetic polymers,"[1] or a collection of "squeezed atoms,"[2-4] or an "electron solid."[5] Mixed approaches have also been attempted.[6,7] Here I adopt the point of view of Kaplan and Glasser,[5] who argued that in a superstrong magnetic field exchange effects[8,9] dominate the behavior at equilibrium of a low-density plasma consisting of electrons and a positive background charge. Consequently, the electron component of the plasma undergoes a Wigner transition[10-12] to an ordered state, which might be called an electron solid.

In this Letter I investigate the occurrence of such a transition as a function of the magnetic field, electron density, and temperature. To the usual variational method[1,2,5] I prefer a differential-equation approach.[3,4] This has the advantage that the mathematical tools of bifurcation theory[13] can be used to compute the transition, without any *ad hoc* assumption,[1,5] about the type of lattice that would arise. The Wigner transition is described as the bifurcation of a spatially periodic solution (solid phase) from the trivial homogeneous solution (plasma phase).[12] Under the usual approximations for an electron gas in a superstrong magnetic field (adiabaticity hypothesis, restriction to the ground Landau level, Hartree-Fock approximation),[3] my treatment of the electrons is exact at zero temperature. Temperature effects are incorporated in a first approximation by performing a trial-function calculation; this avoids the cumbersome task of having to solve a nonlinear integral equation for the electron distribution function.[12] The distribution of positive charge is assumed to be given, rather than arising from the dynamics of the plasma; following a common procedure,[10-12] it will be approximated by a "jellium"—a neutralizing uniform background. The various approximations of the model are discussed briefly at the end of the Letter.

In the Hartree-Fock approximation for an electron gas in a superstrong magnetic field B at temperature T, the electron number density $n(\vec{r})$ can be written as[4]

$$n=\frac{eB}{4\pi^2\hbar^2c}(2mk_BT)^{1/2}I_{-1/2}(\eta), \tag{1}$$

where $I_\alpha(\eta)$ denotes a Fermi-Dirac function,[14] and $\eta(\vec{r})$ is a function related to the electrostatic potential $\varphi(\vec{r})$ and to n. The explicit relationship is obtained from a variational principle.[12] The free energy of the electron gas is written as $F[\eta]=\int nf\,d^3r$, where[4,9]

$$f=k_BT\left(\eta-2\frac{I_{1/2}(\eta)}{I_{-1/2}(\eta)}\right)+\frac{2\pi e\hbar c}{B}n\left\{\ln\left[\pi^2\left(\frac{2\hbar c}{eB}\right)^{3/2}n\right]-\frac{\Lambda}{2}\right\}+e\varphi, \tag{2}$$

Λ being a constant.[15] Minimization of the functional $F[\eta]$, with the constraint $\int n\,d^3r=\text{const}$, yields

$$k_BT\eta+\frac{2\pi e\hbar c}{B}n\left\{2\ln\left[\pi^2\left(\frac{2\hbar c}{eB}\right)^{3/2}n\right]-\Lambda+1\right\}+e\varphi-\mu=0, \tag{3}$$

where μ is a Lagrange multiplier. My model, based on Eqs. (1) and (3), may be thought of as the lowest nontrivial approximation to the dynamics of the electron gas at nonzero temperature.

Poisson's equation $\Delta\varphi = -4\pi e(n-\rho)$ can now be used to eliminate φ from Eq. (3); ρ stands for the number density of positive elementary charge, assumed to be constant. Then, Eqs. (1) and (3) form a system of two nonlinear equations for the electron number density, n, and the trial function, η. Now I introduce the dimensionless quantities

$$\vec{x} = (\lambda^2 a_0)^{-1/3}\vec{r},$$
$$N(\vec{x}) = \pi^3\lambda^2 a_0 n(\vec{r}), \quad R = \pi^3\lambda^2 a_0\rho, \tag{4}$$
$$\tau = 2\hbar^2 k_B T/me^4,$$

where $\lambda = (\hbar c/eB)^{1/2}$ is the electron gyroradius and a_0 is the Bohr radius. My basic system of equations then reads

$$N = \tfrac{1}{4}\pi\tau^{1/2} I_{-1/2}(\eta),$$
$$\Delta\left\{\tfrac{1}{2}\pi^2\tau\eta + 4N\left[\ln\left(\frac{2^{3/2}\lambda}{\pi a_0}N\right) - \frac{\Lambda-1}{2}\right]\right\} = 4\left(\frac{a_0}{\lambda}\right)^{2/3}(N-R); \tag{5}$$

it depends on the three parameters λ, R, and τ.

Equations (5) have the trivial solution

$$\bar{N} = \tfrac{1}{4}\pi\tau^{1/2} I_{-1/2}(\bar{\eta}) = R\,. \tag{6}$$

I am interested in solutions that bifurcate from this homogeneous solution and are periodic. The bifurcation points are determined from the condition that the linearized equations for $N'(\vec{x}) = N(\vec{x}) - \bar{N}$ and $\eta'(\vec{x}) = \eta(\vec{x}) - \bar{\eta}$ have nontrivial solutions. Expressed in terms of the Fourier amplitudes $N'(\vec{k})$ and $\eta'(\vec{k})$, this condition yields the "dispersion relation"

$$-|\vec{k}|^2 g(\lambda,\bar{\eta},\tau) = (a_0/\lambda)^{2/3}, \tag{7}$$

where

$$g(\lambda,\bar{\eta},\tau) = -\frac{\pi\tau^{1/2}}{I_{-3/2}(\bar{\eta})} + \ln\left[\frac{\lambda\tau^{1/2}}{\sqrt{2}a_0} I_{-1/2}(\bar{\eta})\right] + \frac{3-\Lambda}{2}\,. \tag{8}$$

A periodic solution can therefore bifurcate only inside the *bifurcation domain* $D = \{(\lambda,\bar{\eta},\tau) | g < 0\}$.

The Fermi-Dirac functions have been studied in detail, both analytically and numerically,[14] and the behavior of the function $g(\lambda,\bar{\eta},\tau)$ can be followed without difficulty, despite its seemingly complicated appearance. For any given (λ,τ), $g \to \infty$ as $|\bar{\eta}| \to \infty$; g has one minimum at some finite $\bar{\eta}_{\min}(\tau)$, which is independent of λ. Moreover, g is a monotonically increasing function of λ, for any $(\bar{\eta},\tau)$. Knowing this, let us consider the curve $C_0 = \{(\lambda_0,\bar{\eta}_0,\tau_0) | g = 0,\ \partial g/\partial\bar{\eta} = 0\}$, which lies on the boundary of D. Its projection onto the (λ, τ) plane, shown in Fig. 1, will be referred to as the *critical curve*. For any (λ,τ) above the critical curve, $g > 0$ independently of the value of $\bar{\eta}$, and the bifurcation cannot take place. Below the critical curve, there exists an interval H about $\bar{\eta}_0$, such that $g < 0$ if $\bar{\eta} \in H$, and $g > 0$ outside it;

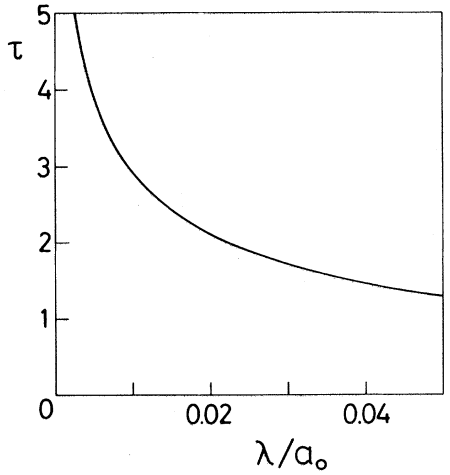

FIG. 1. Critical curve in the (λ,τ) plane.

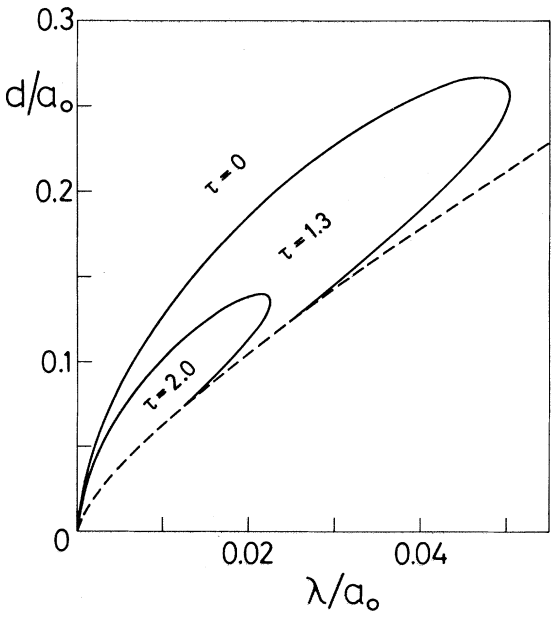

FIG. 2. Bifurcation diagram in the (λ,d) plane, for three values of τ.

this interval collapses to the point $\bar{\eta}_0$ when (λ,τ) tends to (λ_0,τ_0) on the critical curve. Correspondingly, the *bifurcation diagram* at a given temperature (i.e., a section τ = const through the bifurcation domain) has the aspect shown in Fig. 2, for three values of τ. Instead of $\bar{\eta}$, I used the mean interelectron spacing, $d=(3/4\pi\bar{n})^{1/3}$; its explicit relationship to $\bar{\eta}$ follows from Eqs. (4) and (6). At nonzero temperature, the Wigner transition can occur only inside the lobes shown in Fig. 2, which shrink towards the origin as the temperature increases. Higher magnetic fields are needed to maintain long-range order, by increasing the exchange energy,[5] against the disruptive effect of increased thermal agitation at low densities, or of increased Fermi energy at high densities. The limit of zero temperature is obtained by use of the well-known asymptotic expansions of the Fermi-Dirac functions and eliminating $\bar{\eta}$ between Eqs. (6) and (8). The bifurcation domain then expands to cover the entire region above the dashed curve in Fig. 2.

It is sometimes forgotten that the bifurcation diagram does not rigorously coincide with the *phase diagram*, which indicates the domain of existence of the stable solid phase. To see this, let us go back for a moment to the full nonlinear equations (5). Lacking information about the existence, multiplicity, and stability of their solutions, we must resort to a qualitative description in terms of the so-called response diagram of bifurcation theory.[13] In addition to the parameters λ, d, and τ, a solution will be characterized by a set of *order parameters* $\xi=\{\xi_1,\xi_2,\ldots\}$ measuring its departure from the trivial homogeneous solution. The response diagram is a plot of all the solutions in the (λ,d,τ,ξ) space. The possible occurrence of secondary bifurcations and dynamical instabilities, as well as—in our case—the existence of a three-dimensional continuum of bifurcation points and the degeneracy of the bifurcation solutions,[16] could make this picture quite complicated. It can be visualized as a tree growing from the bifurcation domain into the additional dimensions ξ, with possible loops attached, and even disconnected pieces. For any given (λ,d,τ), the physically relevant solution, selected on the basis of a stability criterion (e.g., the minimum of the free energy), corresponds in general to a set of order parameters $\xi_{stab}\neq\{0\}$. The domain of existence of the stable solid phase is obtained by intersecting the response diagram with the surface $\xi=\xi_{stab}(\lambda,d,\tau)$; its size and shape, and even its topology, could be different from those of the bifurcation domain.

At the low densities and temperatures relevant for this problem, the adiabatic assumption and the ground-level restriction are good approximations.[3,4] On the other hand, electron correlations, which enhance the Wigner transition, are sizable,[17] and should be included in a more precise calculation.[18] The trial-function calculation, based on the *Ansatz* of Eq. (1), is justified by the fact that the electron distribution function is similar to the Fermi-Dirac distribution, even when exchange becomes important.[19] At nonzero temperature, corrective terms should be added to the energy, Eq. (2), making Λ a function of η.[19] The fact that such terms tend to be canceled by correlation effects increases the accuracy of the present simplified model, in which both corrections are neglected. The standard assumption of a smeared-out positive charge[10-12] becomes more realistic as the magnetic field increases; however, the existence of an ion lattice can be entirely neglected only in the limit of infinite magnetic field.

The idea that superstrong magnetic fields thread the surface of neutron stars has received new support from the discovery of the 53-keV line in the spectrum of Her X-1,[20] which points to a surface field of 5×10^{12} G. Taking this as a typical value, our calculation sets an upper bound of 3×10^5 K to the temperature at which the Wigner transition can take place, in good agreement with estimates of the surface temperatures of neutron stars, based on their cooling mechanisms.[21] On the other hand, the maximum electron number density at which the bifurcation survives, in the same field but at zero temperature, is 10^{27} cm^{-3}, corresponding to a mass density of 4×10^3 g cm^{-3} for an iron crust. In the same conditions, a variational calculation assuming an electron gas moving in an ion lattice yields a density several times higher.[1,7] This discrepancy could be attributed, trivially, to the difference between the phase and bifurcation diagrams already mentioned; or, more likely, to the oversimplifying assumption of a uniformly distributed positive charge. It appears, however, that condensed matter in the outer crust of pulsars is very close to an electron solid; viz., its properties are dominated by electron exchange and correlations.

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[16]Equation (7) determines the correlation length $l = 2\pi/|\vec{k}|$, but not the wave vector itself. An orthorhombic lattice (Ref. 1) or a two-dimensional lattice of infinte hexagonal rods (Ref. 5) represent two of the many patterns arising from the breaking of Euclidean symmetry onto a smaller group allowing only for discrete translations and rotations. A general selection rule for symmetry breaking at bifurcations is given by D. H. Constantinescu, L. Michel, and L. A. Radicati, J. Phys. (Paris) 40, 147 (1979); see also D. H. Constantinescu, to be published.
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Evidence for Superconducting Effects on the Measured Resistivity Well above T_c for a Type-I Bulk Metal

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A superconducting contribution $\delta\sigma(T)$ to the electrical conductivity $\sigma(T)$ has been observed at temperatures up to about twice the transition temperature $T_c \simeq 1.17$ K for bulk aluminum. The form of $\delta\sigma(T)$ is found to be $\alpha[T_c/(T-T_c)]^n$, with α independent of $\sigma(T)$, in agreement with the theory of superconducting fluctuations. However, significant differences from the theory are found, with α more than two orders of magnitude too large and $n \simeq \frac{3}{4}$ rather than $\frac{1}{2}$.

Superconducting effects in the form of enhanced diamagnetism have been observed[1] at temperatures up to about twice the transition temperature T_c for type-I bulk metals. We report here the first experimental observation of enhanced electrical conductivity for a type-I bulk metal. We find traces of the effect up to about $2T_c$. As was concluded for the enhanced diamagnetism, the enhanced conductivity is attributed to fluctuation superconductivity.

Enhanced conductivity well above T_c has been previously observed[2] for other classes of materials, such as thin films and very dirty bulk samples. However, in contrast to almost all these previous observations, our results show an effect which is more than two orders of magnitude larger than the prediction of Aslamazov and Larkin,[3] Maki,[4] and Thompson.[5] Moreover, we find a temperature dependence of $[T_c/(T-T_c)]^{3/4}$, intermediate between the predicted exponent of $\frac{1}{2}$ for bulk