PHYSICAL REVIEW LETTERS

VOLUME 43 | 2 JULY 1979 | NUMBER 1

Properties of an Associative Algebra of Tensor Fields. Duality and Dirac Identities

Nikos Salingaros and Max Dresden
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 15 March 1979)

An algebra of forms in Minkowski space has been constructed. A multiplication between forms is defined as an extension of the quaternionic multiplications. The algebra obtained is associative with respect to this multiplication of order 16. Duality is expressed as (new) multiplication by a basis element. Vector identities in the algebra lead to a number of new trace identities. A new derivative operator expresses the four Maxwell equations in an especially transparent form.

Many contemporary studies are concerned with the construction of an appropriate algebraic framework for theoretical physics—especially particle physics (see Horwitz and Biedenharn[1] for an impressive effort and an extensive bibliography). The purpose of this paper is to present some new algebraic structures, an algebra, ring, and a group, which allow an alternative and convenient formulation of many laws and equations of physics. The advantages of this formulation are both conceptual and manipulatory; on the former level interesting and important physical notions are described naturally and compactly, while in the latter computational simplifications result from the formulation. It is reasonable to hope that such new formulations will yield new insights.

In this note, the construction of the algebra is briefly outlined; as an example of the conceptual utility, the duality notion is formulated, and as an example of the manipulatory uses, several new identities involving Dirac matrices are obtained. As a by-product, representations are obtained of many groups and algebras important in physics. No proofs are given here; rather, the results are stated and the basic logic is outlined.[2]

Consider a Minkowski space with metric $g^{\mu\mu}=(-1,-1,-1,+1)$ $(x,y,z,=1,2,3;\ t=4)$. Let σ^μ $(\mu=1,2,3,4)$ be the differential *one*-forms in that space. Using the exterior product $\wedge$, construct the 16 basis forms $f_{[\alpha]}$, $\alpha=1,\dots,16$:

$$1,\ \sigma^\mu,\ \sigma^\mu\wedge\sigma^\nu,\ \sigma^\mu\wedge\sigma^\nu\wedge\sigma^\rho,\ \sigma^1\wedge\sigma^2\wedge\sigma^3\wedge\sigma^4=\omega,\quad \mu\neq\nu,\ \mu\neq\nu\neq\rho. \tag{1}$$

Although the exterior multiplication is associative, the basis forms $f_{[\alpha]}$ do not form a group. (There is no inverse.) One can construct a ring by considering expressions such as $\sum c_{[\alpha]}f_{[\alpha]}$, where the $c_{[\alpha]}$ are real scalars. The one-forms σ^μ in the Minkowski space also allow an inner product $(\sigma^\mu,\sigma^\nu)=g^{\mu\nu}$, where $g^{\mu\nu}$ again is the metric. The central point of this paper is the definition of a "new" multiplication (also called "vee" multiplication) between forms. For two one-forms this multiplication, symbolized by $\sigma^\mu\vee\sigma^\nu$, is given as

$$\sigma^\mu\vee\sigma^\nu=\sigma^\mu\wedge\sigma^\nu+g^{\mu\nu}. \tag{2}$$

For n such one-forms, the definition reads

$$\sigma^{\mu_1} \vee \sigma^{\mu_2} \cdots \vee \sigma^{\mu_n} = \sum_{k=0}^{n} \sum_{P} (-1)^P g^{\mu_1\mu_2} g^{\mu_3\mu_4} \cdots g^{\mu_{2k-1}\mu_{2k}} \sigma^{\mu_{2k+1}} \cdots \sigma^{\mu_n}. \tag{3}$$

Here $(-1)^P$ is the signature of the permutation[3]

$$\begin{pmatrix} 1 \cdots n \\ \mu_1 \cdots \mu_n \end{pmatrix}.$$

It follows that the vee product between n one-forms is a linear combination of wedge products of rank n, $n-2, \ldots$. Thus the vee product is an element of the ring. It follows trivally from the definition (2) that (again for a Minkowski metric) $\sigma^\mu \vee \sigma^\nu = \sigma^\mu \wedge \sigma^\nu$, $\mu \neq \nu$ and $\sigma^\mu \wedge \sigma^\mu = g^{\mu\mu}$ (*no* sum). Consequently, the basis $f_{[\lambda]}$, (1), could equally well be written in terms of the vee multiplication as

$$f_{[\lambda]} \equiv \{1, \sigma^\mu, \sigma^\mu \vee \sigma^\nu, \sigma^\mu \vee \sigma^\nu \vee \sigma^\rho, \sigma^1 \vee \sigma^2 \vee \sigma^3 \vee \sigma^4 = \omega\}, \quad \mu \neq \nu, \ \mu \neq \nu \neq \rho. \tag{4}$$

The ring $R(1,3)$ is defined as the set $\sum c_{[\lambda]} f_{[\lambda]}$, with c_λ scalars. It contains sums of differential forms of unequal rank. Clearly the vee product between (any number of) one-forms is an element of $R(1,3)$. Two important results will be noted here: (1) The multiplication is associative. (This can be directly verified from the definitions.) (2) The (16) basic forms $f_{[\alpha]}$, (4), define a *multiplicative group,* called $G(1,3)$ with the vee ($\vee$) multiplications as group operation. (This can also be verified from the definitions but it is better to exploit the duality notion.) It is interesting and a little surprising that the vee multiplication confers this group structure on the forms $f_{[\alpha]}$. It should be stressed that $R(1,3)$ is not a division ring; a sum of forms of unequal rank which occur in $R(1,3)$ has no inverse in $R(1,3)$. However, $R(1,3)$ is "sectionally divisible," each form of definite rank possessing an inverse in $G(1,3)$.

In a four-dimensional space such as the Minkowski space the maximum rank of a wedge product is four. This is not true for the vee product. This circumstance leads to a simplification for the vee product of n factors:

$$\sigma^{\mu_1} \vee \sigma^{\mu_2} \vee \cdots \vee \sigma^{\mu_n} = \sum_P (-1)^P g^{\mu_1\mu_2} \cdots g^{\mu_{n-1}\mu_n} + \sum (-1)^P \sigma^{\mu_1} \wedge \sigma^{\mu_2} g^{\mu_3\mu_4} \cdots g^{\mu_{n-1}\mu_n} + \sum (-1)^P \omega \epsilon^{\mu_1\mu_2\mu_3\mu_4} g^{\mu_5\mu_6} \cdots g^{\mu_{n-1}\mu_n}. \tag{5a}$$

Equation (5a) holds for *n even*; the sums are over all the permutations of the indices $\mu_1, \ldots, \mu_n$. For *n odd* the corresponding formula is

$$\sigma^{\mu_1} \vee \sigma^{\mu_2} \vee \cdots \vee \sigma^{\mu_n} = \sum (-1)^P \sigma^{\mu_1} g^{\mu_2\mu_3} \cdots g^{\mu_{n-1}\mu_n} + \sum (-1)^P \sigma^{\mu_1} \wedge \sigma^{\mu_2} \wedge \sigma^{\mu_3} g^{\mu_4\mu_5} \cdots g^{\mu_{n-1}\mu_n}. \tag{5b}$$

The contact with physics is established by constructing an associative algebra of tensor fields, called $A(1,3)$.[4] A general element of $A(1,3)$ is

$$\hat{\alpha} = F_{(0)} + \sum_\mu F_{(1)}{}^\mu \sigma^\mu + \sum_{\mu,\nu} F_{(2)}{}^{\mu\nu} \sigma^\mu \vee \sigma^\nu + \sum_{\mu,\nu,\rho} F_{(3)}{}^{\mu\nu\rho} \sigma^\mu \vee \sigma^\nu \vee \sigma^\rho + F_{(4)} \omega. \tag{6}$$

Here the F's are the contravariant scalar, vector, and antisymmetric tensor components of rank 2, 3, and 4 in the Minkowski space. The F's are the quantities satisfying physical field equations. (For example, an entity of the type of $F_{(2)}{}^{\mu\nu}$ occurs in the Maxwell equations.) Thus the F's could depend on space-time coordinates, and possibly on internal coordinates (isospin). The collection of 16 components $\{F_0, F_{(1)}{}^\mu, F_{(2)}{}^{\mu\nu}, F_{(3)}{}^{\mu\nu\rho}, F_{(4)}\}$ defines the field alternatively as a vector in the vector space of fields. (This has a striking similarity to a superfield Φ, expanded as 16 components on a Grassmann basis!) It is finally convenient to define tensor *types* $\hat{F}_{(K)}$ of rank K by $\hat{F}_0 =$ scalar, $\hat{F}_{(1)} = F_{(1)}{}^\mu \sigma^\mu$ (vector), $\hat{F}_{(2)} = F_{(2)}{}^{\mu\nu} \sigma^\mu \vee \sigma^\nu =$ tensor or bivector, $\hat{F}_{(3)} = F_{(3)}{}^{\mu\nu\rho} \sigma^\mu \vee \sigma^\nu \vee \sigma^\rho =$ pseudovector, $\hat{F}_{(4)} = F_{(4)} \omega$ pseudoscalar. A general element $\hat{\alpha}$ in $A(1,3)$ can then be written as

$$\hat{\alpha} = \sum_{(K)} S_K \hat{F}_{(K)}, \tag{7}$$

where the S_K are numbers, $\hat{\alpha}$ in general has 16 components. With use of the vee multiplication all the algebraic properties of $A(1,3)$ can be determined. Most important is the observation that this is an *associative* algebra of order 16. The vee multiplication of forms in $R(1,3)$ implies an associative multiplication on the vector space of fields. One can also introduce a scalar norm $N^2(\hat{\alpha}) = \mathrm{Sc}(\hat{\alpha} \vee \hat{\alpha})$ where $\mathrm{Sc}(x)$ means the scalar part of x. However, the algebra $A(1,3)$ is *not* a normed algebra; $N^2(\hat{\alpha})N^2(\hat{\beta})$ *does not equal* $N^2(\hat{\alpha} \vee \hat{\beta})$, nor it is a division algebra. Hence, this associative algebra of order 16 *does not* contradict the Hurwitz-Albert-Frobenius classification theorems, which exclude associative algebras of order >4, since these theorems assume that the algebras are either normed or are division algebras.

In the discussion of differential forms, the notion of the *dual* plays an important role. Similarly in Maxwell theory and Yang-Mills theory, the dual of a field is of special significance. It is therefore important that this dual notion can be expressed directly in terms of the algebraic structure developed. Specifically, if λ^p is a p-form (in the Minkowski space), the dual $\overline{\lambda^p}$ is a $(4-p)$-form, given by

$$\overline{\lambda^p} = (-1)^{1+p(p+1)/2}\omega\lambda^p, \quad \text{with } \overline{\overline{\lambda^p}} = (-1)^{p+1}\lambda^p. \tag{8}$$

From (8), the dual of any tensor type follows directly. A great computational simplification results from the recognition that the basis of forms of $A(1,3)$ can be expressed in terms of the (one-forms) four- and three-dimensional duals of those one-forms and two-forms. Thus a simple (canonical) representation can be given for the elements of $A(1,3)$ in terms of the scalar, vector, and tensor (bivector) types and their duals. With use of these decompositions it is easy to show that there can not exist a self-dual (bivector) tensor field in Minkowski space-time. [This proof is specific to tensor fields in Minkowski space-time; self-dual tensor fields could (and do) exist in a Euclidean space-time.]

The following result indicates the manipulatory power of these algebraic methods. Let $\hat{a} = a^\mu\sigma^\mu$ be a vector type (or tensor type 1) a^μ is a contravariant vector in Minkowski space); then the formulas (5a) and (5b) lead immediately (for even n) to

$$\hat{a}_1 \vee \hat{a}_2 \vee \cdots \vee \hat{a}_n = \sum(-1)^P(a_1\cdot a_2)\cdots(a_{n-1}\cdot a_n) + \sum(-1)^P\hat{a}_1\wedge\hat{a}_2(a_3\cdot a_4)\cdots(a_{n-1}\cdot a_n) + \omega\sum(-1)^P[a_1a_2a_3a_4](a_5\cdot a_6)\cdots(a_{n-1}\cdot a_n). \tag{9}$$

Here $[a_1a_2a_3a_4] = a_1{}^{\mu_1}a_2{}^{\mu_2}a_3{}^{\mu_3}a_4{}^{\mu_4}\epsilon^{\mu_1\mu_2\mu_3\mu_4}$ and $(a_1\cdot a_2) = \sum_\mu a_{1\mu}a_2{}^\mu$, the usual scalar product of two vectors in the Minkowski space.

The relationship between the present algebra (group) and the Dirac algebra may be inferred from these theorems: (i) The group of forms $G(3,1)$ is a representation of the Majorana group M. [The Majorana group is the group formed by real matrices ($\mu = 1,\ldots,4$) ± 1, $\pm\gamma^\mu$, $\pm\gamma^\mu\gamma^\nu$, $\pm\gamma^\mu\gamma^\nu\gamma^\rho$, $\pm\gamma^1\gamma^2\gamma^3\gamma^4$, where the γ satisfy the Dirac anticommutation relations.] Clearly M has order 32. (ii) The group of forms $G(2,3)$ is a representation of the Dirac group $D = M + iM$, of order 64.

From these theorems it follows that any identities, or formal results for vectors $\hat{a} = a^\mu\sigma^\mu$ in $A(1,3)$, are valid for vectors $\not{a} \equiv a_\mu\gamma^\mu$ in the Dirac algebra (a_μ is now indeed the covariant component). This relationship transcribes to the connection formulas

$$\mathrm{Tr}(\not{a}_1\cdots\not{a}_n) = 4\mathrm{Sc}(\hat{a}_1 \vee \cdots \vee \hat{a}_n). \tag{10a}$$

With use of the known properties of the new product (5b) this yields

$$\begin{aligned}&\mathrm{Tr}(\not{a}_1\cdots\not{a}_n) = 4\sum{}_P(-1)^P(a_1\cdot a_2)\cdots(a_{n-1}\cdot a_n), \quad n \text{ even};\\ &\mathrm{Tr}(\not{a}_1\cdots\not{a}_n) = 0, \quad n \text{ odd}.\end{aligned} \tag{10b}$$

Entirely parallel to these results one obtains

$$\mathrm{Tr}(i\gamma^5\not{a}_1\cdots\not{a}_n) = 4\,\mathrm{Sc}(\omega \vee \hat{a}_1\cdots \vee \hat{a}_n), \tag{11a}$$

which gives

$$\begin{aligned}&\mathrm{Tr}(i\gamma^5\not{a}_1\cdots\not{a}_n) = 4\sum(-1)^P[a_1a_2a_3a_4](a_5\cdot a_6)\cdots(a_{n-1}\cdot a_n), \quad n \text{ even};\\ &\mathrm{Tr}(i\gamma^5\not{a}_1\cdots\not{a}_n) = 0, \quad n \text{ odd}.\end{aligned} \tag{11b}$$

It is possible to show in $A(1,3)$ that (n odd)

$$\hat{a}_n \vee \hat{a}_{n-1} \vee \cdots \vee \hat{a}_1 = -\hat{a}_1 \vee \cdots \vee \hat{a}_n + 2\sum(-1)^P \hat{a}_1 (a_2 \cdot a_3) \cdots (a_{n-1} \cdot a_n). \tag{12a}$$

This in the Dirac language yields

$$\not{a}_n \cdots \not{a}_1 = -\not{a}_1 \cdots \not{a}_n + 2\sum(-1)^P \not{a}_1 (a_2 \cdot a_3) \cdots (a_{n-1} \cdot a_n). \tag{12b}$$

Similarly, for n even,

$$\not{a}_n \cdots \not{a}_1 = -\not{a}_1 \cdots \not{a}_n + \tfrac{1}{2}\,\mathrm{Tr}(\not{a}_1 \cdots \not{a}_n) + \tfrac{1}{2}\gamma_5\,\mathrm{Tr}(\gamma_5 \not{a}_1 \cdots \not{a}_n). \tag{13}$$

A host of other identities can be derived using this pattern; some are known (at least in special cases), but others do not appear to be in the literature. For example, from the identity (n even)

$$\sum_{\mu=1}^{4} \sigma^\mu \vee \hat{a}_1 \cdots \vee \hat{a}_n \vee \sigma_\mu = 4\sum(-1)^P (a_1 \cdot a_2)(a_3 \cdot a_4) \cdots (a_{n-1} \cdot a_n) - 4\omega \sum (-1)^P [a_1 a_2 a_3 a_4](a_5 \cdot a_6) \cdots (a_{n-1} \cdot a_n), \tag{14a}$$

it follows that

$$\sum_{\mu=1}^{4} \gamma^\mu \not{a}_1 \cdots \not{a}_n \gamma_\mu = \mathrm{Tr}(\not{a}_1 \cdots \not{a}_n) - \gamma^5\,\mathrm{Tr}(\gamma^5 \not{a}_1 \cdots \not{a}_n). \tag{14b}$$

Although a comparable formula for n odd is known, (14b) appears new; it reduces to a known result for the case $n=4$. (12b) and (13) appear entirely new.

Although strictly speaking outside of the scope of this paper it is interesting to mention that this formalism naturally leads to a derivative operator D. D acting on any form λ is given by $D\lambda \equiv \sigma^\mu \vee \partial_\mu \lambda$. For example, $D\hat{F}_1 \equiv D\sum_\nu a^\nu \sigma^\nu = \partial_\mu a^\nu (\sigma^\mu \vee \sigma^\nu)$. In terms of this operator the four sourceless Maxwell equations assume the elegant form $D\hat{F}_2 = 0$. It is easy to show that $(D \vee D)\lambda = \Box\lambda$, where $\Box$ is the d'Alembert operator. Thus D factors in terms of the vee multiplication.

Further results, including the demonstration that the *inner* automorphisms of $A(1,3)$ leaving the vector type unchanged are just rotations and Lorentz transformations, will be published elsewhere.

Genuine extensions are needed to make the formalism useful in particle physics; the Minkowski space needs to be generalized to a Riemann space, and internal degrees of freedom must be incorporated. The results announced here, apart from having an intrinsic interest, should provide a basis for optimism that such an extension is possible.

This work was supported in part by National Science Foundation Grant No. PHY 76-15328.

---

[1]L. P. Horwitz and L. C. Biedenharn, J. Math. Phys. (N.Y.) 20, 269 (1979).

[2]An entirely parallel construction could be carried out in *any* space as long as one-forms and a scalar product of these forms are defined. The notation used later, $A(1,3)$ [generally $A(p,n)$], refers to the number of plus and minus signs in the metric in the original space. (In later papers the Minkowski space will be replaced by a Riemann space.)

[3]As given by (2), this product (in the case of three dimensions), is isomorphic to the biquaternion product in Clifford's first algebra. Clifford did not generalize this result to n dimensions), [W. K. Clifford, Am. J. Math. 1, 350 (1878)]. Related but distinct products were introduced by E. Caianiello, Nuovo Cimento, Suppl. 5, 171 (1959); E. Caianiello, *Combinatorics and Renormalization in Quantum Field Theory* (Benjamin, Reading, Mass., 1973); D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966). The definition (3) for the vee product of n-forms ($n>2$) has not been considered before.

[4]The algebra called here $A(1,3)$ can be shown to be the Clifford algebra of the space-time forms in a Minkowski space. It should be stressed that the recognition of $A(1,3)$ as the Clifford algebra by itself is by no means sufficient to derive the duality or the Dirac-type identities. To obtain those results (and indeed *all* other results) requires the complete use of the formalism outlined in this paper.