[3]D. A. G. Deacon et al., Phys. Rev. Lett. 38, 897 (1977).
[4]L. R. Elias et al., Stanford High Energy Physics Lab Report No. HEPL-812, 1976 (unpublished).
[5]A. Renieri, Comitato Nazionale per l'Energia Nucleare–Frascati Report No. 77-30, 1977 (unpublished); L. R. Elias et al., Stanford High Energy Physics Lab Reports No. HEPL-824 and No. HEPL-831, 1978 (unpublished).
[6]J. M. J. Madey, Ph.D. thesis, Stanford University, 1970 (unpublished), p. 150; J. Bridges et al., *Fermilab Electron Cooling Experiment Design Report* (U. S. GPO, Washington, D. C., 1978).
[7]W. B. Colson, *Physics of Quantum Electronics* (Addison-Wesley, Reading, Mass., 1977), Vol. 5.
[8]W. H. Louisell et al., Phys. Rev. A 18, 655 (1978).
[9]H. Kildal and R. L. Byer, Stanford Microwave Report No. MW-83, 1970 (unpublished).
[10]R. H. Pantell et al., IEEE J. Quantum Electron. 4, 905 (1968).
[11]N. M. Kroll and W. A. McMullin, Phys. Rev. A 17, 300 (1978); F. A. Hopf et al., Opt. Commun. 18, 413 (1976).
[12]See Bridges et al., Ref. 6.

"Convective" Loss-Cone Instability Is Absolute

M. J. Gerver
Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14853
(Received 11 December 1978)

The high-frequency "convective" loss-cone mode in a mirror-confined plasma is shown to be absolutely unstable as a result of wave reflection from the mirror throats, provided that the plasma length is greater than an axial wavelength. Critical lengths for stability are only a few ion gyroradii for fusion parameters, much smaller than previous estimates. This result places serious limitations on the design of mirror fusion reactors, and precludes the possibility of a linearly stable reactor with empty loss cone.

The high-frequency convective loss-cone (HFCLC) mode was first considered in 1966 by Post and Rosenbluth,[1] who showed that it would seriously limit plasma confinement in a magnetic mirror machine more than a few hundred ion gyroradii in length, but that, in the absence of wave reflection at the mirror throats, it would not grow to significant levels in shorter machines. Since then, a number of possible reflection mechanisms have been considered. Aamodt and Book[2] pointed out that even if the wave has no turning points where $k_\parallel(z)=0$ (z being the axial coordinate and $k_\parallel$ the axial wave number in the WKB approximation), there will still be a small amount of reflection, of order $\exp(-k_\parallel a)$, where a is the scale length of the mirror throats. This reflection can be interpreted as being due to the slight corrections to the WKB approximation. Equivalently it may be viewed as being due to reflection of the wave from complex turning points at which $k_\parallel(z)=0$, where $k_\parallel(z)$ has been extended analytically into the complex z plane. Since the wave grows by a factor of $\exp(-\operatorname{Im}k_\parallel L)$ as it traverses the length of the machine ($\operatorname{Im}k_\parallel < 0$ indicates growth), an absolute instability will occur if $|\operatorname{Im}k_\parallel|L > k_\parallel a$. Aamodt and Book[2] considered those modes with the highest convective growth rates, viz. $\omega \simeq \omega_{pi}$ and $k_\perp \lambda_D \simeq 1$, and for these modes $|\operatorname{Im}k_\parallel| \ll k_\parallel$; so absolute instability does not occur, although the critical length is somewhat reduced.

Other reflection mechanisms include nonlocal wave reflection due to coherent bouncing of electrons,[3] and reflection from turning points due to ion cyclotron resonances.[4] With any reflection mechanism, the axial wavelength is reduced by electromagnetic (finite β) effects,[5] and this can reduce the critical length. Taking all of these effects into account, recent estimates[6] of the critical length have been about 50 gyroradii, considerably less than the original estimate,[1] but still quite tolerable for a mirror fusion reactor.

In this Letter, I consider only the reflection mechanism discussed by Aamodt and Book.[2] When longer-wavelength modes ($k_\perp\lambda_D \ll 1$, $\omega \ll \omega_{pi}$) are considered, then $\operatorname{Im}k_\parallel \simeq k_\parallel$, and absolute instability occurs, even though the local convective growth rates are lower than for the modes considered by Aamodt and Book.[2] The critical length for these modes should be roughly one axial wavelength. For fusion parameters and for the worst modes, the critical length is only a few ion gyroradii. If the length of the mirror machine is less than a few ion gyroradii, then the radius of the plasma must also be less than a few ion gyroradii, and the plasma will be subject to the drift-cone instability,[1] unless the loss cone is partly filled in with warm plasma.[7] Hence, any mirror-fusion plasma with empty loss cone must be unstable.

I consider a Vlasov plasma with straight-line ion orbits and strongly magnetized electrons, and assume electrostatic perturbations of frequency ω and perpendicular wave number $k_\perp$. This model is appropriate for $\Omega_i \ll \omega \ll \Omega_e$ and $\Omega_e/v_e \gg k_\perp \gg \Omega_i/v_i$, where Ω_i, Ω_e, v_i, and v_e are the ion and electron cyclotron frequencies and thermal velocities; and $\omega_{pe}^2 \ll k_\perp^2 c^2$. I also assume $k_\parallel v_e \ll \omega$, so that electron (and ion) Landau damping can be neglected, and $k_\perp \gg k_\parallel$; these assumptions are justified later. The ions are assumed to have perpendicular velocity distribution

$$f_i(v_\perp) = v_\perp^2 \exp(-v_\perp^2/v_i^2),$$

appropriate to a loss-cone plasma. The Vlasov-Poisson equation for the perturbed potential $\varphi(z)$ is then

$$\frac{d}{dz}\frac{\omega_{pe}^2(z)}{\omega^2}\frac{d\varphi}{dz} + k_\perp^2\varphi + \frac{\omega_{pi}^2(z)\omega}{k_\perp v_i^3} Z''\left(\frac{\omega}{k_\perp v_i}\right)\varphi = 0, \tag{1}$$

where Z'' is the second derivative of the Fried-Conte plasma dispersion function.[8] For $\omega \ll k_\perp v_i$, $Z''(\omega/k_\perp v_i) \simeq -2\pi^{1/2} i$. For a definite model, I take $\omega_{pe}^2(z) \simeq \omega_{pe0}^2 z^3/L^3$ near the mirror throat, located at $z=0$; here L is the scale length of the plasma, and ω_{pe0}^2 is ω_{pe}^2 at the midplane. This z^3 dependence is appropriate for a collisional ion distribution[9]; a collisionless distribution would be even more unstable. Equation (1) may then be written in dimensionless form

$$\frac{d}{dS} S^3 \frac{d\Phi}{dS} - i\alpha e^{3i\theta}(S^3 + iS_0^3 e^{-i\theta})\Phi = 0, \tag{2}$$

where $S \equiv z/L$, $\Phi(S) \equiv \varphi(z)$, $\alpha \equiv 2\pi^{1/2}(m_e/m_i)(|\omega|^3 L^2/k_\perp v_i^3)$, $S_0^3 \equiv \frac{1}{2}\pi^{-1/2} k_\perp^3 v_i^3/|\omega|\omega_{pi0}^2$, and $\theta \equiv \arg\omega$, i.e., $\omega = |\omega| e^{i\theta}$. Equation (2) has three WKB turning points, at $S = S_0 \exp(-\pi i/6 - i\theta/3)$, and at $\exp(2\pi i/3)$ and $\exp(-2\pi i/3)$ times this value.

The global behavior of $\Phi(S)$ may be seen by plotting the Stokes lines and anti-Stokes lines for Eq. (2). Defining

$$K_\parallel^2(S) = -i\alpha e^{3i\theta}[1 + (iS_0^3/S^3)e^{-i\theta}], \tag{3}$$

the WKB approximation to the potential is

$$\Phi(S) = C_1 \exp[+i\int^S K_\parallel(S')\,dS'] + C_2 \exp[-i\int^S K_\parallel(S')\,dS'],$$

where C_1 and C_2 are constant within a given Stokes region.[10] The pattern of Stokes lines (solid lines) and anti-Stokes lines (dashed lines) is shown in Fig. 1 for real positive ω (i.e., $\theta = 0$). A solution which is purely left-going just to the right of the origin (corresponding to complete absorption at the mirror throat) will still be purely left-going (at a lower amplitude) along the anti-Stokes line going between the origin and the turning point at $S_0 \exp(-\pi i/6)$, labeled S_t in the figure. In the vicinity of S_t, where $k_\parallel^2(S) \simeq -3\alpha S_0^3(S - S_t)S_t^{-4}$ and Eq. (2) may be approximated by Airy's equation, the solution must be

$$\Phi(S) = \mathrm{Ai}\,((3\alpha/S_0)^{1/3} \exp(-4\pi i/9)(S - S_t)), \tag{4}$$

without any Bi component. Then the solution has left-going and right-going components of equal amplitude along the anti-Stokes line which originates at S_t and goes up and to the right. Analytically continuing the solution to the real axis, we see that anywhere to the right of the point (labeled P) where this anti-Stokes line crosses the real axis, the right-going component has greater amplitude than the left-going component. In other words, if a left-going wave with real frequency ω is launched to the right of point P, it will re-

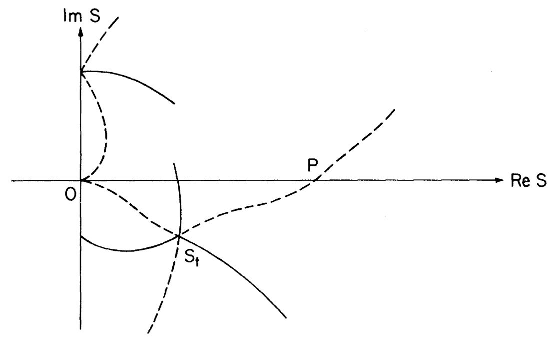

FIG. 1. Anti-Stokes lines (dashed) and Stokes lines (solid) for $\theta = 0$ (purely real ω).

turn as a right-going wave of greater amplitude.

To construct an unstable normal mode, assume that the other mirror throat is far to the right of point P, and let θ be slightly less than $\pi/6$. The pattern of Stokes lines for this case is shown in Fig. 2. The procedure for constructing a solution with absorbing boundary conditions at the origin is similar to the previous case, but with $S_t \simeq S_0 e^{-\pi i/4}$. If θ were exactly $\pi/6$, then the anti-Stokes line going to the right from S_t would be asymptotically parallel to the real axis, remaining below it. Because θ is slightly less than $\pi/6$, the anti-Stokes line crosses the real axis at a slight angle, far to the right of S_t. If θ is adjusted so that the anti-Stokes line crosses the real axis at the midplane, then this anti-Stokes line will coincide with the anti-Stokes line originating from the turning point S_t' associated with the other mirror throat. We will then have a normal mode with $\mathrm{Im}\,\omega > 0$, i.e., an absolute instability, whenever

$$\int_{S_{t'}}^{S_t} K_{\parallel}(S)\,dS = \pi(n+\tfrac{1}{2}), \qquad (5)$$

for integer n large enough so that the WKB approximation is valid.

Several assumptions have been made in constructing this unstable mode:

(A) It was assumed that the other mirror throat (which occurs at S of order unity) is far to the right of point P in Fig. 1. This implies $S_0 \ll 1$, which means $\omega\omega_{pi0}^2 \gg k_\perp^3 v_i^3$.

(B) The integer n appearing in Eq. (5) must be much greater than unity for the WKB approximation to be valid. Since $|S_{t'} - S_t|$ is of order unity, and $K_\parallel^2(S) \simeq \alpha$ over most of the path of integration, according to Eq. (3), this implies $\alpha^{1/2} \gg 1$, or

$$L \gg k_\perp^{1/2} v_i^{3/2} |\omega|^{-3/2} (m_i/m_e)^{1/2}. \qquad (6)$$

This inequality simply states that the machine must be longer than an axial wavelength, a result found previously by Berk, Pearlstein, and Cordey.[4]

(C) To neglect electromagnetic effects, $\omega_{pe}^2 \ll k_\perp^2 c^2$ is required in the vicinity of the turning points. Since $\omega_{pe}^2 \simeq \omega_{pe0}^2 S_0^3$ in the vicinity of the turning points, this implies $\omega/k_\perp v_i \gg T_i/m_e c^2$.

(D) The straight-line ion-orbit approximation requires $\omega \gg \Omega_i$ and $k_\perp v_i \gg \Omega_i$.

(E) The electrons were assumed to be strongly magnetized, implying $\omega \ll \Omega_e$ and $k_\perp v_e \ll \Omega_e$.

(F) The approximation $Z''(\omega/k_\perp v_i) \simeq -2\pi^{1/2} i$ requires $\omega \ll k_\perp v_i$.

(G) To neglect Landau damping, it is necessary that $k_\parallel \ll \omega/v_e$ everywhere between the turning points. In the WKB approximation, $k_\parallel = K_\parallel/L$, and from Eq. (3), $K_\parallel \lesssim \alpha^{1/2}$ everywhere between the turning points, so Landau damping can be neglected if $\alpha^{1/2} \ll L\omega/v_e$, or $\omega/k_\perp v_i \ll T_i/T_e$. [Note that Landau damping *does* take place in a small region very close to the origin ($|S| \ll S_0$) even when this last condition is satisfied, and this justifies the use of absorbing boundary conditions at the origin.]

In a mirror machine $T_i > T_e$, so that conditions E and G are automatically satisfied if conditions A and F are. In any fusion reactor $T_i < m_e c^2$ and $\omega_{pi0} \gg \Omega_i$; so there will be some range of ω and $k_\perp$ where A, C, D, and F are all satisfied. This range is shown schematically in Fig. 3. Unstable modes exist provided Eq. (6) is satisfied for some value of ω and $k_\perp$ in this range. The right-hand side of Eq. (6) is smallest in the upper right-hand corner of this range, where conditions A and F are both marginally satisfied, viz., $\omega \simeq \omega_{pi0}$, and $k_\perp \simeq \omega_{pi0}/v_i$. The length needed for any mode to be unstable may then be found from Eq. (6):

$$L \gtrsim (m_i/m_e)^{1/2} v_i/\omega_{pi0} = \rho_i \Omega_e/\omega_{pe0}, \qquad (7)$$

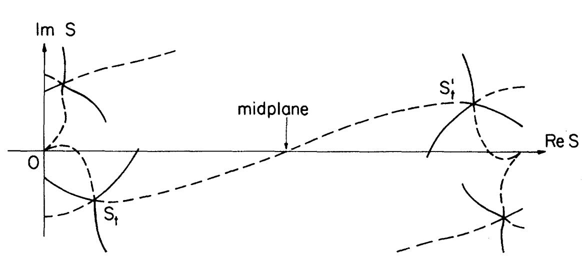

FIG. 2. Anti-Stokes lines (dashed) and Stokes lines (solid) for a normal mode; θ is slightly less than $\pi/6$.

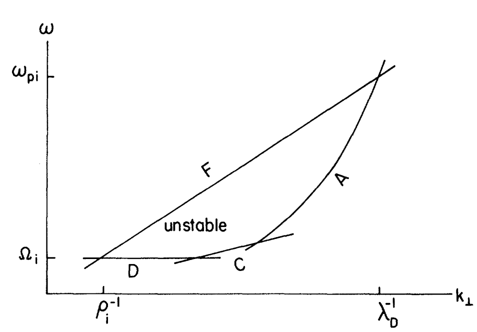

FIG. 3. The unstable region of the ω-$k_\perp$ plane, where conditions A, C, D, and F are all satisfied (not drawn to scale).

where the ion gyroradius $\rho_i \equiv v_i/\Omega_i$. Since $\omega_{pe0} > \Omega_e$ is necessary for an economical fusion reactor, the critical length is on the order of an ion gyroradius or less.

A number of ways to avoid this instability suggest themselves, and have been examined. The results are summarized here:

(1) As pointed out above, the worst modes [those whose axial wavelength, given by the right-hand side of Eq. (6), is shortest] have very high frequency, $\omega \simeq \omega_{pi}$, and perpendicular wave number, $k_\perp \lambda_D \simeq 1$; these modes may saturate nonlinearly at a low level, and be relatively harmless. The longer-wavelength modes (which might be expected to be harmful to plasma confinement) have much greater critical lengths for instability. For $\omega \simeq \Omega_i$ and $k_\perp \rho_i \simeq 1$, the critical length would be $L/\rho_i \simeq (m_i/m_e)^{1/2}$. Unfortunately, when finite-β effects are included,[5] the axial wavelength is shortened by a factor of $(1+\omega_{pe}^2/k_\perp^2 c^2)^{1/2}$, or about $\beta^{1/2}(m_i/m_e)^{1/2}(k_\perp \rho_i)^{-1}$. Since an economic fusion reactor requires β not too much less than unity, the critical length is $L/\rho_i \simeq \beta^{-1/2} \simeq 1$, even for the modes with $\omega \simeq \Omega_i$ and $k_\perp \rho_i \simeq 1$.

(2) Reflection can be reduced by plasma outside the mirror throats. If the plasma density goes as $\omega_{pe}^2(z) = \omega_{pe0}^2(\Delta + z^3/L^3)$ around the mirror throat at $z=0$, then there will be significant stabilization only when $\Delta \simeq 1$, i.e., when there is almost as much plasma outside the mirror throats as inside.

(3) Warm plasma in the loss cone will have a stabilizing effect, but will not stabilize the worst modes, which have $\omega/k_\perp \simeq v_i$, until the loss cone is almost completely full.

(4) Even if Eq. (7) is accepted as an order of magnitude estimate, there may be numerical factors which make the critical length considerably greater than an ion gyroradius. To examine this possibility, critical lengths have been determined by numerically solving a differential equation similar to Eq. (1), but including electromagnetic effects, as well as the dependence of the ion temperature and the magnetic field on z. For parameters typical of a mirror fusion reactor ($T_i = 100$ keV, $\beta = 0.6$ at the midplane, mirror ratio $R=3$) the critical length was $17.2\rho_i$, with the worst mode having axial-mode number $n=1$.

The HFCLC instability would not be expected to be absolutely unstable in present mirror experiments, such as 2XIIB. In 2XIIB, the radial scale length is so small that the loss cone must be almost completely filled with warm plasma to avoid the drift-cone instability. With the loss cone filled, *all* loss-cone instabilities are stabilized.

In future experiments and in proposed mirror fusion reactors, the HFCLC instability should be important. For the fusion reactor plasma considered above, absolute stability of these modes with no warm plasma would require $L/\rho_i < 17.2$. But this would imply a radial scale length R_p so small ($R_p/\rho_i \lesssim 9$) that the drift-cone mode would be unstable. Thus a linearly stable fusion plasma with empty loss cone is impossible. If $R_p/\rho_i = 9$, then 1% warm plasma will stabilize the drift-cone mode. Hence a stable fusion plasma is possible if the loss cone is filled with 1% warm plasma.

Requiring 1% warm plasma puts severe, but not impossible, constraints on the design of a mirror fusion reactor. In particular, 1% warm plasma might be tolerable if some kind of end stoppering is used, and it might be tolerable in the end cell of a tandem mirror reactor.

This research was inspired by discussions with E. Ott, who made many helpful suggestions. The treatment of Stokes lines is largely a result of suggestions of H. L. Berk. Useful discussions with L. D. Pearlstein, R. N. Sudan, J. M. Finn, G. R. Smith, and D. C. Watson are also acknowledged. This work was supported by the U. S. Department of Energy Contract No. EY-76-S-02-3170-1.

[1]R. F. Post and M. N. Rosenbluth, Phys. Fluids 9, 730 (1966).

[2]R. E. Aamodt and D. L. Book, Phys. Fluids 9, 143 (1966).

[3]H. L. Berk, C. W. Horton, M. N. Rosenbluth, D. E. Baldwin, and R. N. Sudan, Phys. Fluids 11, 365 (1968).

[4]H. L. Berk, L. D. Pearlstein, and J. G. Cordey, Phys. Fluids 15, 891 (1972).

[5]J. D. Callen and G. E. Guest, Phys. Fluids 14, 1588 (1971).

[6]D. E. Baldwin, Rev. Mod. Phys. 49, 317 (1977).

[7]H. L. Berk and M. J. Gerver, Phys. Fluids 19, 1646 (1976).

[8]B. F. Fried and S. Conte, *The Plasma Dispersion Function* (Academic, New York, 1961).

[9]H. L. Berk, Phys. Fluids 14, 1036 (1971).

[10]W. H. Furry, Phys. Rev. 21, 360 (1947).