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Paramagnetism for Nonrelativistic Electrons and Euclidean Massless Dirac Particles

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We construct the manifold of zero-energy eigenstates for a nonrelativistic spin-$\frac{1}{2}$ particle moving in a plane in an external magnetic field $B(\vec{x})=\Sigma_{n=0}^{N}\lambda_n(\vec{x}-\vec{c}_n)$, with $\{\lambda_n\}$ and $\{\vec{c}_n\}$ arbitrary reals and $\{k_n\}$ positive integers. For a given B the ground state is infinitely degenerate and the manifold of eigenfunctions is parametrized by a point in $R^{2(2k\max+1)}$. For such B's we prove paramagnetism with arbitrary external potential $V(\vec{x})$.

Paramagnetism for nonrelativistic electrons is well known. It means decrease of the free energy as a function of the external magnetic field or positivity of the susceptibility. A full quantum mechanical proof is not yet available. Textbook discussions are based on perturbation theory in the external field, semiclassical considerations, or very simple model Hamiltonians.

In this note paramagnetism for zero temperature, i.e., decrease of the ground-state energy of an electron in an external magnetic field $\vec{B}$ and an external potential V, is proved. The magnetic field is assumed to have only a z component B, depending on the variables x and y. On the way we prove a simple but remarkable fact: The ground-state energy for very general magnetic fields but vanishing external potential is infinitely degenerate. The ground states can be constructed explicitly.

Consider the two-dimensional Schrödinger Hamiltonian

$$H(\vec{a})=[\vec{\sigma}\cdot(\vec{p}-\vec{a})]^2=(\vec{p}-\vec{a})^2-B\sigma_z\geq 0, \qquad (1)$$

with $\vec{p}=(-i\partial_x,-i\partial_y)$, $B=\partial_x a_y-\partial_y a_x$, and σ_i the Pauli matrices.[1] $H(\vec{a})$ describes the motion of a charged particle with spin $\frac{1}{2}$ and no anomalous magnetic moment[2] in an external magnetic field B. This corresponds, in three dimensions, to magnetic fields that have (x,y) dependence and point in the z direction. At the same time, $H(\vec{a})$ is the square of the Euclidean two-dimensional Dirac operator with zero mass, in an external magnetic field. The infimum of the spectrum of $H(\vec{a})$, the ground-state energy if this point belongs to a bound state, will be denoted by $\inf\{\mathrm{Spec}H(\vec{a})\}$.

For $B=$const, $H(\vec{a})$ was solved in 1930 by Landau.[3] Here, we shall construct the zero-energy eigenstates of $H(\vec{a})$ for a large class of magnetic fields [see Eq. (3)] that includes polynomials in x^2+y^2 and their translates. These solutions saturate the bound zero in Eq. (1) *independently of* $\vec{a}$.[4] For $\vec{a}$ fixed, the zero energy is infinitely degenerate and the manifold of normalizable eigenstates is parametrized by a point in $R^{2(2k_{\max}+1)}$. $k_{\max}=\max\{k_n\}$ is determined by B [see Eq. (3)].

These results are used below to prove paramagnetism with external potential [Eq. (7)].

Paramagnetism for spinning particles has two physical interpretations. In the context of Euclidean field theory it means positivity of the effective interaction Lagrangian (obtained by integrating out the fermionic degrees of freedom).[5] In quantum mechanics, it is related to the phenomenon of enhanced binding in magnetic fields, e.g., the existence of He^-.[6]

The feature that makes the zero-energy problem solvable for a large class of fields is that the eigenvalue equation can be linearized[7] in $\vec{a}$. Consider

$$\vec{\sigma}\cdot(\vec{p}-\vec{a})\psi=0. \tag{2}$$

Evidently, $H(\vec{a})\psi=0$. The operator $\vec{\sigma}\cdot(\vec{p}-\vec{a})$ is linear in $\vec{a}$ and by superposing B fields for which (2) is solvable one obtains again fields for which (2) is solvable (with suitable products of ψ's). We shall give below the explicit solutions ψ for (2) with $B(x)=(x^2+y^2)^N$, N being a nonnegative integer.[8] By superposition we are thus able to solve (2) for any field B that can be written as a polynomial[9] in x^2+y^2 and translates of such polynomials, i.e.,

$$B(\vec{x})=\sum_{n=0}^{N}\lambda_n(\vec{x}-\vec{c}_n)^{2k_n}. \tag{3}$$

Let $z=x+iy$, $\bar{z}=x-iy$, $A=a_x+ia_y$, $\bar{A}=a_x-ia_y$, and finally $\psi=(\psi_+,\psi_-)$. Equation (2) reads

$$(2i\partial_z+\bar{A})\psi_+=0, \quad (\partial i\partial_{\bar{z}}+A)\psi_-=0. \tag{4}$$

Let $A=i\lambda[2(N+1)]^{-1}\bar{z}^N z^{N+1}$, λ real. Since $B(x)=-i(\partial_z A-\partial_{\bar{z}}\bar{A})$, we have $B(x)=\lambda(x^2+y^2)^N$; substituting in (3) and (4) gives

$$\psi_+=\kappa_+\exp\left[\frac{\lambda}{4(N+1)^2}(z\bar{z})^{N+1}+\mathcal{P}_+(z)\right], \tag{5}$$

$$\psi_-=\kappa_-\exp\left[\frac{\lambda}{4(N+1)^2}(z\bar{z})^{N+1}+\mathcal{P}_-(z)\right]. \tag{6}$$

$\kappa_{+,-}$ are constants and $\mathcal{P}_{+,-}$ are arbitrary analytic functions. For $\lambda>0$, ψ is square integrable if $\kappa_+=0$ and $\mathcal{P}_-$ is a polynomial of degree $2N+1$. Similarly $\kappa_-=0$ for $\lambda<0$. A polynomial of degree $2N+1$ has $2N+2$ independent coefficients of which the constant term fixes the normalization of ψ. Thus the zero energy is infinitely degenerate and its states are labeled by a point in a $2(2N+1)$-dimensional real vector space.[7] In the constant-field case, $N=0$, the vector space is a plane whose points designate the center of rotation of the Landau orbit. In analogy, one may say that in the general case a point in the $2(2N+1)$-dimensional vector space gives the "position and shape" of the particle.

In the framework of Euclidean field theory the zero-energy states can be interpreted as fermionic pseudoparticles.

This result is interesting from several points of view:

(i) In the case $B=\text{const}$ the infinite degeneracy is explained to be a consequence of translation invariance. We see that at least as far as the ground state is concerned, an infinite degeneracy holds irrespective of the translation invariance of the magnetic field.

(ii) The existence of a zero mode[10] is related to a general paramagnetic inequality (conjectured in Ref. 5) for spin-$\frac{1}{2}$ particles.[10] A special case[11] of this inequality is

$$\inf\{\text{Spec}[H(\vec{a})+V]\}\leq\inf\{\text{Spec}[H(\vec{a}=0)+V]\}, \tag{7}$$

where V is an arbitrary potential.[12] The existence of zero-energy eigenfunctions of $H(\vec{a})$ proves (8) for the two-dimensional case with $V=0$. In fact, for this case the equality sign holds because $H(\vec{a})$, being the square of self-adjoint operators, is positive.[13]

(iii) The stability of $\inf\{\text{Spec}H(\vec{a})\}$ for a large class of fields, singular with respect to $H(\vec{a}=0)$, is a remarkable fact.

(iv) That the bottom of the spectrum is essential (i.e., not an isolated, finitely degenerate eigenvalue) is relevant to the absence of magnetic bottles for fermions. For spin-0 particles, on the other hand, there are fields B such that $(\vec{p}-\vec{a})^2$ has a purely discrete spectrum.[14] What distinguishes particles without spin from particles with spin is that spinless particles have a zero-point energy proportional to $|B|$ while spinning particles have no zero-point energy.[15]

Lieb[16] has proven (7) for $B=\text{const}$. By extending his method and using the explicit form for the zero-energy eigenstates we can prove (7) for any field B of the form (3). The proof holds both in two and three dimensions [in three dimensions $\vec{B}=(0,0,B)$].

Assume E_0 to be a discrete ground state of[17] $H(a=0)+V$:

$$[H(a=0)+V]f=E_0 f. \tag{8}$$

Take B as in (3) and assume $\lambda_N > 0$. Write

$$\psi_{\{d\}} = \sum_{n=1}^{N} \varphi_{n,d_n}, \quad \varphi_{n,d_n} = \exp\left[-\frac{\lambda_n}{4(k_n+1)^2} |z - c_n|^{2(k_n+1)} + d_n (z - c_n)^{k_n+1} - \frac{(k_n+1)^2}{\lambda_n} d_n \bar{d}_n \right]. \tag{9}$$

$\psi_{\{d\}}$ is a zero-energy solution for all $\{d\}$. Note that it is square integrable over the $2N$-dimensional $\mathrm{Im}\{d\}, \mathrm{Re}\{d\}$ space and that

$$\nabla |\varphi_{n,d_n}|^2 = \partial_{\bar{z}} |\varphi_{n,d_n}|^2 = -\frac{\lambda_n}{2(k_n+1)} (\bar{z} - \bar{c}_n)^{k_n} \partial_{d_n} |\varphi_{n,d_n}|^2. \tag{10}$$

By direct computation, the expectation value[18] is

$$\langle f\psi_{\{d\}}, [H(a) + V] f\psi_{\{d\}} \rangle = E_0 \| f\psi_{\{d\}} \|^2 + \alpha(\{d\}), \tag{11}$$

where

$$\alpha(\{d\}) = -2 \int \nabla |f|^2 \nabla |\psi_{\{d\}}|^2 \, dx\, dy \tag{12}$$

(note that α is real). But, by (10) and the integrability of $|\alpha|$ with respect to the $\{d\}$ variables,

$$\int \alpha(\{d\}) \prod_{i=1}^{N} d(\mathrm{Im} d_i)\, d(\mathrm{Re} d_i) = 0. \tag{13}$$

It follows that $\alpha(\{d\})$ cannot be positive for all values of $\{d\}$. By the minimum-maximum principle this proves (7).

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[1]For dimensions larger then three, $\vec{\sigma}$ should be replaced by $\vec{\gamma}$ matrices with $\{\gamma_\mu, \gamma_\gamma\} = 2\delta_{\mu\nu}$, $\mu, \nu = 1, \cdots, d$.

[2]This is crucial. If the modulus of the magnetic moment is smaller than 1, one expects diamagnetism. If it is larger than 1, $H(a)$ will, in general, be unbounded below for fields that diverge at infinity.

[3]L. D. Landau, Z. Phys. 64, 629 (1930), and in *The Collected Papers of L. D. Landau*, edited by D. Ter Haar (Pergamon, New York, 1965).

[4]Compare with B. Simon and I. Herbst, "Some remarkable examples in Eigenvalue Perturbation Theory" (to be published).

[5]H. Hogreve, R. Schrader, and R. Seiler, "A conjecture on the Spinor Functional Determinant" (to be published).

[6]Independently conjectured in the constant-field case in J. Avron, I. Herbst, and B. Simon, "Schrödinger Operators with Magnetic Fields. I. General Interactions" (to be published), and Phys. Rev. Lett. 39, 1068–1070 (1977).

[7]This is reminiscent of pseudoparticle solutions of Yang-Mills theories. See, e.g., A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 22, 503 (1975) [JETP Lett. 22, 245–247 (1975)]. The situation there is distinguished by vector potentials that belong to the Lie algebra of, say, SU(2) and so is noncommutative. In addition, attention is focused on solutions with finite energy and so with $B \to 0$ at infinity.

[8]As a differential equation (2) can be solved for a much larger class of A's. This choice of B makes the discussion of normalizability of ψ particularly simple (see below).

[9]Formally the most general $B(x)$ which can be obtained by superposition is

$$B(x) = \sum_{n=0}^{\infty} \int d\mu(\vec{c})\, (\vec{x} - \vec{c})^{2kn}.$$

We are indebted to J. Bellissard for this remark.

[10]For related questions see R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3397 (1976).

[11]For a spinless Bose system the converse is true and diamagnetism has been proven by B. Simon for arbitrary $\vec{a}$, mutual interaction potential, and external potentials [Phys. Rev. Lett. 36, 1083–1084 (1976)]. This result can be extended to the situation where $\vec{a}$ is an external Yang-Mills field. It leads to a diamagnetic inequality for the renormalized Euclidean functional determinant: R. Schrader and R. Seiler, Commun. Math. Phys. 61, 169 (1978).

[12]There are related paramagnetic inequalities for the partition function and the functional determinant; see Ref. 5 for details. The paramagnetic inequality for the functional determinant is known in the following cases: in one dimension; in two dimensions it follows from an analysis of the Schwinger model; in three dimensions it has been shown by D. Bridges, J. Fröhlich, and

E. Seiler, "On the Construction of Quantized Gauge Fields I. General Results" (to be published); in four dimensions the result is known for the case of constant field strength. Analogous inequalities for the relativistic situation have been shown by J. Schwinger, Phys. Rev. 93, 615 (1954).

[13]By stability theorems for the essential spectrum, the equality sign in (7) extends to a larger class of B's than the one considered here. In particular, it extends to smooth B's with falloff at infinity.

[14]For example, $B(x)=x^2+y^2$ (more generally, any $B \to \infty$ at infinity); see Ref. 6 for details. Note that Ref. 9 and the positivity of $H(\vec{a})$ imply that $H(\vec{a})$ has no discrete ground state.

[15]Indeed for spin-0 particles

$$\inf\{\mathrm{Spec}(p-a)^2\} \geq \inf_{\vec{x}\in R^3} |\vec{B}(\vec{x})|.$$

[16]The theorem of Lieb appears in an appendix to Ref. 11.

[17]For example, if $V \in L^{3/2}(R^3)+L^\infty(R^3)$, $a_\mu \in L_{\mathrm{loc}}{}^2(R^3)$; this involves no loss of generality. Let $V_\epsilon = V + \epsilon \vec{x}^2$. $H(0)+V_\epsilon$ has a discrete ground state that converges to $\inf\{\mathrm{Spec}[H(0)+V]\}$. Define $H(a)+V_\epsilon$ by the Friedrichs extension on C_0^∞. The inequality (7) is then proven by passing to the limit $\epsilon = 0$.

[18]We use the natural identification of $f\psi_{\{d\}}$ with the spinor-valued function $(0,\psi_{\{d\}})$.

Implications of General Covariance and Maximum Four-Dimensional Yang-Mills Gauge Symmetry

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General covariance and maximum four-dimensional Yang-Mills gauge symmetry lead to these results: (1) Gravity is characterized by a dimensionless constant $F \sim 10^{-19}$; (2) the Newtonian force is always attractive; (3) space-time has a torsion; (4) gravitational spin-force between two protons is about 10^{19} times stronger than the corresponding Newtonian force. A possible experimental test is discussed.

The idea of gauge symmetry has been developed to obtain simple and elegant spin-1 fields by Yang and Mills.[1] When this idea was extended to spin-2 fields such as gravity, the dynamics of interactions becomes extremely complicated.[2-4] This may be due to the conceptual bondage of the conventional approach which postulates the Riemannian metric tensor as basic field variables. The Yang-Mills–type gauge symmetry for gravity has been studied by many physicists.[3-6] The results are stimulating but not completely satisfactory. Also, previous formulations of gravity involves a dimensional coupling constant, which leads to serious divergences in higher orders.

In this Letter, we explore a different approach in which Yang-Mills gauge fields, associated with maximum four-dimensional symmetry (i.e., the de Sitter group), are regarded as basic dynamical fields and the metric tensor is postulated to be a function of gauge fields. The physical motivation is to combine the two basic principles, i.e., the Yang-Mills gauge symmetry and the Einstein general covariance, in such a way that the formulation of gravity, including fermions, involves a small *dimensionless* coupling constant and agrees with experiments. Furthermore, the dynamics of the gravitational interaction and the maximum four-dimensional gauge symmetry are interlocked in the same way as that in electromagnetism. Thereby, serious divergences could be reduced and other problems[4] can be resolved as well.

We stress that the de Sitter group is used only as the gauge group so that the dynamics of interaction between fields are uniquely specified by such a gauge symmetry. One should not interpret the de Sitter–group operators to be the translational and rotational operators of physical space-time. In other words, the physical space may not be the same as the de Sitter space.[7]

We first observe that in analogy with the electric force, the gravitational force can be written as $-F_1F_2/r^2$, where $F_1 = G^{1/2}m_1$ and $F_2 = G^{1/2}m_2$ are dimensionless (for $c=\hbar=1$). This suggests that the de Sitter group is natural for the gauge group of gravity because it involves a length L. The matrix representation[6,8] Z_A of the de Sitter-group generator is given by

$$Z_A = (Z_i, Z_a) = (\gamma_i/2L,\ i\gamma_j\gamma_k/2),$$