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Superflow in Restricted Geometries

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We propose a "competing barrier model" for nucleation of quantized vortices in small channels. The results are compared to experiments on decay of persistent currents, critical velocities, onset temperatures, and the effective superfluid density at onset.

For a number of years experimentalists have tried to use the Iordanskiĭ-Langer-Fisher[1,2] (ILF) theory of fluctuation dissipation for superflow in an unbounded region (intrinsic nucleation) to fit experiments in restricted geometries. Several years ago, two of us (DR)[3] pointed out the necessity of considering a permanent barrier ΔE for flow in a restricted geometry which is present, irrespective of any superflow. For a toroidal channel of circular cross section with zero circulation, DR point out that the probability of nucleating to the wall a vortex of one sign is equal to that for a vortex of opposite circulation. However, when there is a superflow present, the energy barriers for nucleation of the two types of vortex are not the same: For a large superflow the process associated with one type of vortex will completely dominate the other, while for slower flows the nucleation of both types of vortex have to be considered. This is the "competing-barrier model" of nucleation.

In this Letter we show that the competing-barrier model will qualitatively explain experiments on decay of persistent currents as reported, for example, by Hallock and co-workers[4] and Kojima *et al.*[5] We confine ourselves to the broad issues using a simplified one-dimensional nucleation model, neglecting the geometrical differences among various sorts of porous materials and thin films.

The nucleation probability, P, per unit time can be written

$$P = f\exp(-\Delta F/kT), \tag{1}$$

where f is the temperature-dependent attempt frequency discussed and tabulated by DR and ΔF is the barrier height. For the ILF model, ΔF is due to the superflow alone. Here, $\Delta F = \Delta E$ when the superflow $v_s = 0$ and the critical momentum is then p_c. With a flow, the $\vec{p}\cdot\vec{v}_s$ interaction shifts the barrier and (considering only one dimension) we have $\Delta F = \Delta E \pm p_c v_s$ so that (1) becomes

$$P = 2f\exp(-\Delta E/kT)\sinh(p_c v_s/kT). \tag{2}$$

Equation (2) shows that the dimensionless quantity V ($\equiv p_c v_s/kT$), the ratio of ordered flow energy to fluctuation energy, is important.

Suppose the superflow takes place in a toroidal geometry containing n candidates per unit length for nucleation; then (cf. Ref. 3)

$$dv_s/dt = -nP\kappa, \tag{3}$$

where $\kappa \equiv h/m$. Equation (3) can be written nondimensionally as

$$dV/d\tau = -\sinh V \tag{4}$$

by introducing the dimensionless time $\tau = \nu_0 t$ and the nucleation rate ν_0:

$$\nu_0 = \nu\exp(-\Delta E/kT), \quad \nu = 2n\kappa f p_c/kT. \tag{5}$$

Equation (4) has the solution

$$V = \ln[(1+e^{-\tau}\tanh V_0/2)/(1-e^{-\tau}\tanh V_0/2)], \tag{6}$$

where V_0 is the value of V at $\tau = 0$. The character of (6) depends on the ranges of V_0 and τ.

For infinitesmal superflows, such as in "Nth sound" ($N = 2, 3, 4$), $V_0 \to 0$ and (6) gives for all τ

$$V \cong \ln[(1 + \tfrac{1}{2}V_0 e^{-\tau})/(1 - \tfrac{1}{2}V_0 e^{-\tau})] \cong V_0 e^{-\tau}. \qquad (7)$$

Thus all small superflows decay exponentially.

For $V_0 \to \infty$ and $V \to \infty$ simultaneously, $e^{-V} \cong \exp(-V_0) + \tau/2$ so that

$$V \cong V_0 \text{ (small } \tau), \qquad (8a)$$

$$V \cong \ln(2/\tau) \text{ (large } \tau), \qquad (8b)$$

and the dividing case occurs at τ_L obtained by equating the two estimates in (8):

$$\tau_L = 2\exp(-V_0). \qquad (9)$$

For fixed τ (> 0), V is independent of V_0 in the limit $V_0 \to \infty$:

$$V \cong \ln\{(1 + e^{-\tau})/(1 - e^{-\tau})\} = \ln\coth(\tfrac{1}{2}\tau),$$

so that

$$V \cong \ln(2/\tau) \text{ (small } \tau), \qquad (10a)$$

$$V \cong 2e^{-\tau} \text{ (large } \tau), \qquad (10b)$$

and the dividing case occurs at τ_E obtained by equating the two estimates in (10):

$$\tau_E \cong 1. \qquad (11)$$

The dimensionless times τ_L and τ_E are fundamental to our discussion: τ_E is universal and sets the lifetime of all superflows, while τ_L depends on V_0. The flow observed depends on the magnitude of τ. For $\tau \ll \tau_L$ the flow is almost steady; for $\tau_L \ll \tau \ll \tau_E$ the flow shows logarithmic behavior; for $\tau \gg \tau_E$ the flow decays exponentially. The number of decades of logarithmic behavior is given by $\log(\tau_E/\tau_L) \cong 0.43V_0 - 0.37$.

Figure 1 shows several examples of (6), the time evolution of finite superflows. In particular, the flow for $V_0 = 15$ has $\log\tau_L = -6.2$. Larger initial flows are independent of V_0 at $\log\tau = -6.2$, and and experimenter observing at this τ would term $V_0 = 15$ the "saturated critical velocity." Hence, the condition at time τ for a saturated critical velocity is given by $\tau = \tau_L$ and thus the notion of a saturated critical velocity depends crucially on the time of observation.

An experimenter using Nth sound as a probe for evidence of superfluidity would observe nothing for times larger than τ_E. Thus the condition for "onset of superfluidity" is given by $\tau = \tau_E$.

In order to make numerical calculations, we need estimates of ΔE, n, and f. Imagine a channel containing packed powder with a mean open dimension d. A vortex stretched between two grains will have an energy $E_A = (\rho_s \kappa^2 d/4\pi)\ln(d/a)$, where a is the vortex core parameter. When the line is moved into a semicircle, it will just touch the next grain, and then its energy is $E_c = (\rho_s \kappa^s d/4)\ln(d/a)$. The "barrier" is given by $\Delta E = E_c - E_A$ and $p_c = \frac{1}{2}\rho_s \kappa\pi d^2$. To order one, we adopt for simplicity

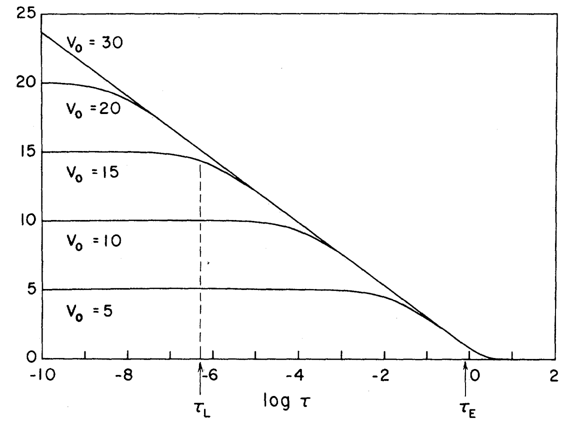

FIG. 1. Plot of Eq. (6) assuming $kT/p_c = 4$ and various values of the initial dimensionless velocity V_0. The dimensionless time showing the beginning of exponential decay τ_E is common to all curves. τ_L is shown for $V_0 = 15$. Initial flows greater than 15 have the same velocity near $\log\tau = -6$; initial flows less than 15 are "steady" for increasing periods as V_0 decreases.

$$\Delta E = (\rho_s \kappa^2 d/4\pi)\ln(d/a), \qquad (12a)$$

$$p_c = \rho_s \kappa d^2. \qquad (12b)$$

In the same spirit, since the preexponential factors are not important, we assume that n corresponds to one trapped vortex betwen each pair of grains, $n = 1/d$, while a constant value $f = 10^8$ sec^{-1} will suffice. For films we imagine the substrate contains a distribution of trapped vortex lines pinned between protuberances on the substrate. When a trapped line of length of order d moves into semicircle, it will just touch the free surface, forming a vortex of energy E_c as before. Thus the estimates of ΔE, f, and n may be retained for simplicity.

When a saturated current decays according to (8b), the slope of the decay is $dV/d\ln\tau = -1$, or

$$dv_s/d(\log t) = -kT\ln(10)/p_c. \qquad (13)$$

Kojima *et al.*[5] observed a decay of 0.63% per decade of a saturated persistent current with $v_s = 67.7$ cm/sec at $t = 1$ sec, $T = 1.3$ K, and $d = 170-$

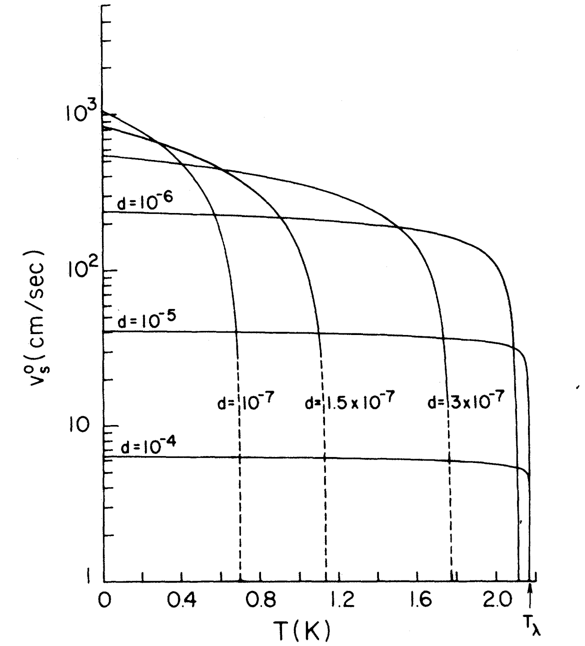

FIG. 2. Saturated velocities obtained from Eq. (14) at $t = 100$ sec for various channel sizes. Flows near onset will show decay as $V_0 \to 1$ (indicated by dashed lines).

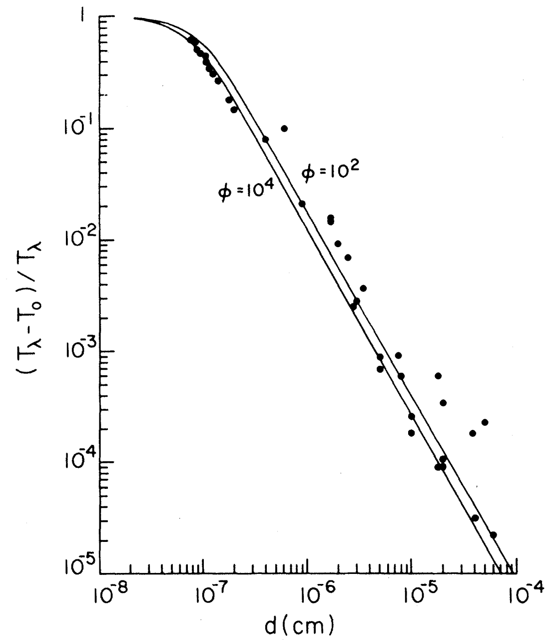

FIG. 3. Onset temperatures T_0 obtained from various channel sizes. Data from many sources.

325 Å. Thus $dv_s/d(\log t) = -6.3\times10^{-3}\times67.7$ or, by (13), $kT/p_c = 0.185$. For $d = 250$ Å we find, using (12b), that $kT/p_c = 0.208$ in good agreement with observation.

The condition that an initial velocity leads to a saturated flow is $\tau = \tau_L$, or $V_0 = \ln(2/\tau)$. Defining v_s^0 as the initial velocity corresponding to V_0, we find from (5) that $\tau = \tau_L$ becomes

$$v_s^0 = \Delta E/p_c - (kT/p_c)\ln(\nu t/2). \tag{14}$$

The first term $\Delta E/p_c = (\kappa/4\pi d)\ln(d/a)$ does not contain temperature explicitly and is a Feynman critical velocity.[6] By itself, it is appropriate only at very low temperatures, since the second term substracts from it, reducing the observed critical velocity. The results shown in Fig. 2 are in order-unity agreement with published results. They show the critical velocity approaching zero near the onset temperature $T_0 < T_\lambda$, to be discussed next.

The condition for the onset of superfluidity is $\tau = \tau_E$. If we express our observation time in terms of frequency $\varphi = t^{-1}$, this condition by (5) is $\nu_0 = \varphi\tau_E$ or

$$[\ln(d/a)/\ln(\nu/\varphi\tau_E)]\,(\rho_s d/T_0) = 4\pi k/\kappa^2, \tag{15}$$

which must be solved by iteration since ν and a are themselves temperature dependent. We show in Fig. 3 the results of T_0 as a function of d compared with the results of many experiments, taking typical values of $\varphi = 10^2$–10^4 Hz. The general agreement is quite satisfactory—below about 0.6 K our results are questionable since the corresponding films are very thin. T_0 increases weakly with φ.

Recently Nelson and Kosterlitz[7] showed that for two-dimensional superfluids the ratio of superfluid mass per unit area near the transition temperature T_0 is given by the exact result,

$$\rho_s(T_0^-)/T_0 = 8\pi k/\kappa^2. \tag{16}$$

Bishop and Reppy[8] and Rudnick[9] have shown that (16) gives a good account of their experiments over a wide range of temperatures and thicknesses identifying $\rho_s(T_0^-)$ with $\bar{\rho}_s d$. We can write (15) in the form (16) by adopting an effective superfluid density at onset:

$$\bar{\rho}_s = \rho_s\{2\ln(d/a)/\ln(\nu/\varphi\tau_E)\}. \tag{17}$$

We illustrate the behavior of $\bar{\rho}_s$ in Fig. 4: It has the usual property of vanishing at $T = 0$ and $T = T_\lambda$. For comparison one can show data quoted by Rudnick derived from third-sound measurements on thin films[9] and by Bishop and Reppy from measurements of $\rho_s(T_0^-)$.[8] Values of $\bar{\rho}_s$ were deduced

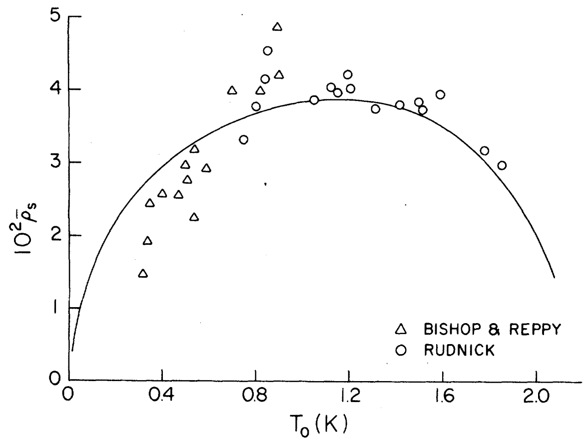

FIG. 4. The effective superfluid density at onset ρ_s calculated from Eq. (17) and compared with experiments of Rudnick (Ref. 9) and Bishop and Reppy (Ref. 8).

assuming the relationship between onset thickness d in atomic layers and T_0 is given by $T_0=\beta(d-\alpha)$ with $\beta\simeq 1.5$ K per layer and $\alpha\simeq 1.4$ layers for $T_0<1$ K.

The works of Nelson and Kosterlitz,[7] Huberman, Myerson, and Doniach,[10] Ambegaokar, Halperin, Nelson, and Siggia,[11] and Myerson[12] address the problem of two-dimensional superfluidity with emphasis on the behavior near the transition. Reference 10 discusses a two-dimensional depairing model and Refs. 10 and 12 give a decay form which fits some of the film data quite well.[4] Reference 11 discusses a vortex depairing and recombination model and provides a detailed discussion of the experiments of Bishop and Reppy.[8] The present theory assumes three dimensions and requires a permanent barrier ΔE. It attempts to include behavior far from critical. Films so thin that the nucleation mechanism discussed above could not operate would have to be treated differently: In particular, the thickness of the film must be considerably greater than the healing length in the present model. We shall address these concerns more fully in a forthcoming article.

Evidence supporting the decay shape shown in Fig. 1 and the dependence of ν_0 and d and T will be presented elsewhere by D. Ekholm and R. B. Hallock, to whom we are indebted for experimental cooperation during the development of this theory. We are also grateful to J. Reppy and I. Rudnick for discussions of their experiments, and to B. Huberman, S. Doniach, E. Siggia, and V. Ambegaokar for the discussions of their theories.

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