Analytical Theory of Drift Waves and Drift-Alfvén Waves in Tokamaks

Y. C. Lee
Department of Physics, University of California, Los Angles, California 90024

and

Liu Chen
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540
(Received 3 November 1978; revised manuscript received 10 January 1979)

The collisionless drift wave and drift-Alfvén wave instabilities in a sheared magnetic field are considered using the complex transformation technique of Antonsen. It is shown that no unstable bounded eigensolutions for the finite-β radial eigenmode equations exist. We also present a more general technique based on the wave-flux conservation and find, furthermore, that the convective amplification of drift waves in the electrostatic limit can be anomalously large, which could lead to enhanced transport in tokamaks.

The recent discovery[1,2] that the collisionless electrostatic drift wave in a sheared magnetic field in slab geometry is absolutely stable has been reinforced by the elegant analytical proof of Antonsen.[3] More recently, Tsang *et al.*[4] have indicated through extensive numerical calculations that the finite-β modified drift modes as well as the shear Alfvén modes are similarly not unstable in the absolute sense. Analytical treatments of finite-β modified drift waves have been attempted by Catto *et al.*[5] and by Chen *et al.*[6] Their treatments, however, are basically perturbative in nature and, hence, cannot determine the stability definitively. As to the shear Alfvén branch, only approximate analytical solutions exist.[7] In this Letter we first extend Antonsen's method to both finite-β modified drift modes and shear Alfvén modes. We demonstrate absolute stability in both cases.

In slab geometry ignoring the compressional component of the perturbed magnetic field, we can write the finite-β radial eigenmode equations[5,6] as

$$\rho_s^2(d^2/dx^2-k_y^2)\varphi=F(x)(\varphi-\Omega x_c A_{\parallel}/x), \qquad (1)$$

and

$$(v_A\rho_s/c)^2(d^2/dx^2-k_y^2)A_{\parallel} = (\Omega x_c/x)F(x)(\varphi-\Omega x_c A_{\parallel}/x); \qquad (2)$$

where $F(x)=(1-1/\Omega)(1+\xi_e Z_e)-(x/\Omega x_s)^2$, $\Omega=\omega/\omega_{*e}$, $\xi_e=\Omega x_e/|x|$, $x_e=\omega_{*e}/k_{\parallel}'v_e$, $k_{\parallel}'=k_y/L_s$, $x_s=\omega_{*e}/k_{\parallel}'c_s$, $c_s^2=T_e/m_i$, $\rho_s=c_s/\omega_{ci}$, $x_c=\omega_{*e}/k_{\parallel}'c$, $Z_e=Z(\xi_e)$ where Z is the plasma dispersion function, and the rest of the notations are standard. We wish to solve Eqs. (1) and (2) subject to the outgoing/evanescent boundary conditions. Thus, the asymptotic ($|x|\to\infty$) behaviors

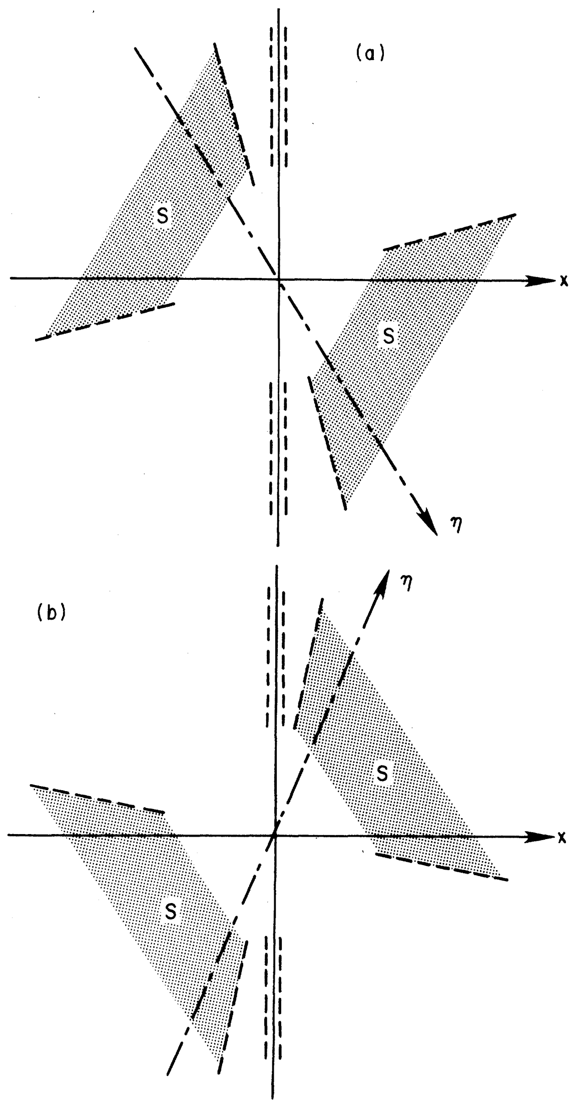

FIG. 1. Asymptotic anti-Stokes lines (dashed lines) for (a) $\mathrm{Re}\Omega>0$ and (b) $\mathrm{Re}\Omega<0$. Also shown are the real η axis (dash-dotted lines) and the subdominant regions (S).

of both φ and $A_\parallel$ will be described by linear combinations of $\exp(-ix^2/2\Omega x_s\rho_s)$ and $\exp(-k_y|x|)$.

Following Antonsen,[3] we first assume that the unstable eigenmodes with $\mathrm{Im}\,\Omega>0$ exist. It follows that both φ and $A_\parallel$ are subdominant (i.e., $|\varphi|$, $|A_\parallel|\to 0$ as $|x|\to\infty$) along the real $-x$ axis. Now if we allow x to be analytically continued into the complex x plane, then for the case $\mathrm{Im}\,\Omega$, $\mathrm{Re}\,\Omega>0$, the anti-Stokes lines lie in the second and fourth quadrants as shown in Fig. 1. Therefore, the line $x=-i\Omega\eta$, where η is real, lies entirely within the subdominant regions as long as $\mathrm{Im}\,\Omega>0$, regardless of the sign of $\mathrm{Re}\,\Omega$.[3] Hence, upon making the transformation[3] $x=-i\Omega\eta$, Eqs. (1) and (2) become

$$-\rho_s^2(\Omega^{-2}d^2/d\eta^2+k_y^2)\varphi=F(\eta)(\varphi-ix_cA_\parallel/\eta), \quad (3)$$

and

$$-(v_A\rho_s/c)^2(\Omega^{-2}d^2/d\eta^2+k_y^2)A_\parallel =(ix_c/\eta)F(\eta)(\varphi-ix_cA_\parallel/\eta), \quad (4)$$

where

$$F(\eta)=(1-1/\Omega)G(\eta)+\eta^2/x_s^2$$

and

$$G(\eta)=1+i(x_e/\eta)Z(ix_e/\eta)\geq 0.$$

Multiplying Eqs. (3) and (4) by $\Omega^2\varphi^*$ and $\Omega^2A_\parallel^*$, respectively, and integrating from $\eta=-\infty$ to $\eta=\infty$ (where both φ and $A_\parallel$ vanish) yield a quadratic equation for Ω,

$$\Omega^2A_1-\Omega A_2-A_3=0. \quad (5)$$

Here, $A_1, A_3>0$ and $A_2\geq 0$ are Ω-independent quadratic forms given by

$$A_1=k_y^2\rho_s^2\langle|\varphi|^2+|v_AA_\parallel/c|^2+[G(\eta)+\eta^2/x_s^2]|\varphi-ix_cA_\parallel/\eta|^2\rangle,$$

$$A_2=\langle G(\eta)|\varphi-ix_cA_\parallel/\eta|^2\rangle,$$

$$A_3=\rho_s^2\langle|d\varphi/d\eta|^2+(v_A/c)^2|dA_\parallel/d\eta|^2\rangle,$$

where $\langle\dots\rangle$ denotes $\int_{-\infty}^{\infty}d\eta(\dots)$. Equation (5) can be formally solved to give

$$\Omega=[A_2\pm(A_2^2+4A_1A_3)^{1/2}]/2A_1.$$

Hence, $\mathrm{Im}\,\Omega=0$ which contradicts the assumption that the mode is unstable and thereby disproves the existence of such modes.

The preceding analysis does not rule out the possibility of marginally stable modes ($\mathrm{Im}\,\Omega=0$). Theoretically, the question of marginal stability is interesting because there seems to be some numerical indications[1,2,8] that for sufficiently weak shear and finite $k_y^2\rho_s^2$, the electrostatic drift wave becomes marginally stable. However, because of the numerical uncertainties inherent in any numerical code, this question must be answered analytically.

Assuming that marginally stable modes ($\mathrm{Im}\,\Omega=0$) do exist, one can immediately construct from Eqs. (3) and (4) the following Wronskian identity:

$$\frac{d}{d\eta}\left[\left(\frac{v_A}{c}\right)^2\left(A_\parallel\frac{dA_\parallel^*}{d\eta}-A_\parallel^*\frac{dA_\parallel}{d\eta}\right)+\varphi\frac{d\varphi^*}{d\eta}-\varphi^*\frac{d\varphi}{d\eta}\right]=0. \quad (6)$$

Since Eq. (6) is valid everywhere along the real η axis, it is also valid asymptotically, $|\eta|\to\infty$. Defining

$$\Gamma(\eta)=i[(v_A/c)^2(A_\parallel dA_\parallel^*/d\eta-A_\parallel^*dA_\parallel/d\eta)+\varphi d\varphi^*/d\eta-\varphi^*d\varphi/d\eta],$$

from Eq. (6) we see that $\Gamma(\eta)$ must be a real constant; i.e., the wave flux is conserved. Asymptotically, φ and $A_\parallel$ should consist of subdominant solutions only, i.e.,

$$\varphi=\alpha_\pm\eta^{-1/2}\exp(i\Omega\eta^2/2x_s\rho_s)+i\beta_\pm(x_c/\eta)\exp(\pm i\Omega k_y\eta) \quad (7a)$$

and

$$A_\parallel=\beta_\pm\exp(\pm i\Omega k_y\eta)+i\alpha_\pm(c/v_A)^2x_c\eta^{-3/2}\exp(i\Omega\eta^2/2x_s\rho_s), \quad (7b)$$

for $\eta\to\pm\infty$. Note that in Eqs. (7a) and (7b) we have kept terms which are of order $1/\eta$ higher than the leading terms in the asymptotic series so as to emphasize that the coupling between φ and $A_\parallel$ decreases rapidly with $|\eta|$. These higher-order terms do not contribute to the wave flux $\Gamma(\eta)$ in the asymptotic limit, where we find

$$\Gamma(\pm\infty)=\pm2[(v_A/c)^2\Omega k_y|\beta_\pm|^2+(\Omega/x_s\rho_s)|\alpha_\pm|^2]. \quad (8)$$

Equation (8) clearly contradicts the constancy of the wave flux Γ unless $\alpha_\pm$ and $\beta_\pm$ are all exactly equal to zero. Thus, we have proved that, within the approximations implied by Eqs. (1) and (2), exact marginally stable eigenmodes cannot exist.

We note that, while Antonsen's method is rather novel, it works only in the case of cold-fluid ions. We now present a more general technique[9] which has much wider applicability. For simplicity let us consider the *electrostatic* case but with *warm kinetic* ions. Here, Eq. (1), with $A_\parallel = 0$, retains the same form except that $F(x)$ is replaced by $G(x, \lambda_i)$, where

$$G(x, \lambda_i) = -\frac{(\tau/\lambda_i^2)(1+\lambda_i/\Omega\tau)(1+\xi_i Z_i) + (1-1/\Omega)(1+\xi_e Z_e)}{(1+\lambda_i/\Omega\tau)\xi_i Z_i}. \tag{9}$$

Here, $\tau = T_e/T_i$, $\xi_i = \Omega x_i/|x|\lambda_i$, $x_i = \omega_{*e}/k_\parallel' v_i$, $Z_i = Z(\xi_i)$, and λ_i $(0 \leq \lambda_i \leq 1)$ is the kinetic-ion parameter. Thus, $\lambda_i = 0$ and 1 correspond to the cases of cold-fluid ions and warm kinetic ions, respectively. As λ_i is increased from 0 to 1, the eigenmode frequencies, $\Omega(\lambda_i)$, would then evolve continuously from the $\lambda_i \to 0^+$ limit. As discussed above, we can use Antonsen's method to prove that, in the $\lambda_i \to 0^+$ limit, the eigenmodes are damped; i.e., $\mathrm{Im}\,\Omega(\lambda_i \to 0^+) < 0$.[10] Furthermore, the preceding analysis on the absence of marginally stable modes in the cold-ion limit can be readily extended to show that, for the present case and the entire range of λ_i $(0 \leq \lambda_i \leq 1)$, there again exists no eigenmode with $\mathrm{Im}\,\Omega = 0$. Thus, $\Omega(\lambda_i)$ never crosses the real-Ω axis and the eigenmode remain damped.

We now address the problem of convective instabilities. For simplicity, we will consider only the purely electrostatic case and assume cold ions. We accomplish this by formulating the problem as a scattering one. Hence, we look for solutions with the following boundary conditions:

$$\varphi = x^{-1/2}\exp(ix^2/2\Omega\rho_s x_s) + R(\Omega)x^{-1/2}\exp(-ix^2/2\Omega x_s\rho_s), \quad x \to \infty,$$

and

$$\varphi = T(\Omega)x^{-1/2}\exp(-ix^2/2\Omega x_s\rho_s), \quad x \to -\infty, \tag{10}$$

where $R(\Omega)$ and $T(\Omega)$ would be interpreted as reflection and transmission coefficients, respectively, and we consider only real values for Ω. It is possible to prove[11] that in a sectorial region bounded by two adjacent anti-Stokes lines there is an uniform asymptotic expansion, whose leading term is a linear combination of the two independent solutions $\varphi_{1,2}$ of the equation

$$[\rho_s^2 d^2/dz^2 - k_y^2\rho_s^2 - 1 + 1/\Omega + (z/\Omega x_s)^2]\varphi = 0. \tag{11}$$

Using the relation between parabolic cylinder functions, we obtain

$$\varphi_1(a, z) = \varphi_1(-a, -iz)\exp[i(\theta + \pi/4)], \tag{12a}$$

$$\varphi_2(a, z) = (2\cosh\pi a)^{-1/2}[\varphi_2(-a, iz)\exp(\pi a + i\pi/4 + i\theta) + \exp(i3\pi/4)\varphi_1(-a, -iz)]; \tag{12b}$$

where $a = (\Omega x_s/2\rho_s)(1/\Omega - 1 - k_y^2\rho_s^2)$ and $\theta = \arg[\Gamma(\frac{1}{2} - ia)]$. φ_1 and φ_2 are the subdominant and dominant solutions to Eq. (11), respectively. We have assumed $\Omega > 0$, but similar relations can be obtained when $\Omega < 0$. Equations (12a) and (12b) enable us to relate the scattering problem along the real-x axis to that along the real-η axis for $\Omega > 0$ $(z = -i\Omega\eta)$. If we define the scattering boundary conditions along η to be

$$\varphi = \eta^{-1/2}\exp(-i\Omega\eta^2/2x_s\rho_s) + r(\Omega)\eta^{-1/2}\exp(i\Omega\eta^2/2x_s\rho_s), \quad \eta \to \infty, \tag{13a}$$

and

$$\varphi = \tau(\Omega)\eta^{-1/2}\exp(i\Omega\eta^2/2x_s\rho_s), \quad \eta \to -\infty, \tag{13b}$$

then $r(\Omega)$ and $\tau(\Omega)$ are related to $R(\Omega)$ and $T(\Omega)$ by

$$R(\Omega) = [r(\Omega) + i]\exp(\pi a), \tag{14a}$$

and

$$T(\Omega) = \tau(\Omega)\exp(\pi a). \tag{14b}$$

Along the real-η axis the scattering problem is greatly simplified, now that the potential is real. For $k_y^2\rho_s^2 < 1/\Omega - 1$, there exist two pairs of real turning points.[8] The inner turning points at $\eta = \pm E$ are determined primarily by the electron Z function and the other ones at $\pm P$ are the usual Pearlstein-Berk ion-sound turning points determined quite accurately as $\pm\Omega x_s(1/\Omega - 1 - k_y{}^2\rho_s{}^2)^{1/2}$. If we ignore the outer turning points, then, since the wave is propagating inside the inner turning points, the eigenvalue is determined by the quantization conditions

$$\int_{-E}^{E} Q^{1/2}(\Omega, \eta)\, d\eta = (n + \tfrac{1}{2})\pi, \quad n = 0, 1, 2\ldots, \qquad (15)$$

where $Q(\Omega, \eta)$ is the real potential. Because of the presence of outer turning points, the wave cannot remain trapped and, therefore, no real-Ω eigensolution is possible. However, if the tunneling factor $\exp(-\int_E^P Q^{1/2}\, d\eta)$ is small, then it is possible to consider the eigenvalue Ω as determined by Eq. (15) to be a resonance. For real Ω sufficiently far from resonance, $r(\Omega) + i = O(\exp(-\int_E^P Q^{1/2}\, d\eta))$ and so $R(\Omega) = O(1)$; in this case there is no significant convective amplification. On the other hand, for Ω very close to resonance, $|r(\Omega) + i| = O(1)$ and both the reflection and transmission coefficients are of order $\exp(\pi a)$ which is assumed to be large. Thus, we can identify $\exp(\pi a)$ as the convective amplification factor, which can be estimated using the data of Ref. 8. Taking $m_i/m_e = 1837$, $L_s/L_n = 50$, and $k_y{}^2\rho_s{}^2 = 0.5$, we find the resonance frequency to be $\Omega \simeq 0.55$. We thus have $\pi a \simeq 13.74$ and the amplification factor is exponentially large, $O(10^6)$![12]

In conclusion we have shown that there are no unstable drift modes and drift-Alfvén modes of either parity and that the convective amplification of the electrostatic drift modes can become exponentially large. The obvious implication of the latter result is that we may expect anomalous transport of a plasma confined in a sheared magnetic field due to the convectively unstable drift modes.

This work was supported by the National Science Foundation Grant No. PHY-77-12873, and the U. S. Department of Energy, Contracts No. EY-76-C-02-3073, No. EY-76-S-03-0034-PA128, and No. W-7405-ENG-48.

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[12]This enormous amplification indicates that the system has eigenmodes with negligibly small damping rates, which cannot be distinguished from marginally stable modes by numerical and WKBJ analyses.

Effect of Beam Scattering on Plasma Heating with Relativistic Electron Beams

J. D. Sethian[(a)] and C. A. Ekdahl[(b)]
Laboratory of Plasma Studies, Cornell University, Ithaca, New York 14850
(Received 20 December 1978)

A fully ionized plasma column was heated with a relativistic electron beam. Experiments in which plasma heating was dominated by the beam-plasma two-stream interaction demonstrated a scaling of energy deposition with the beam-to-plasma density ratio and the beam angular scatter in accordance with a theorectial model of the two-stream instability. Beam-to-plasma energy-transfer efficiencies exceeding 25% and plasma electron temperatures of 600 eV were observed.

The availability of high-power relativistic-electron-beam (REB) generators has stimulated interest in their use for the rapid heating of plasma targets.[1,2] Because energy transfer from the REB to the plasma by means of Coulomb collisions is negligible for the plasma densities and