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Proposed Test for Complex versus Quaternion Quantum Theory

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If scattering amplitudes are ordinary complex numbers (not quaternions) then there is a universal algebraic relationship between the six coherent cross sections of any three scatterers (taken singly and pairwise). A violation of this relationship would indicate either that scattering amplitudes are quaternions, or that the superposition principle fails. Some experimental tests are proposed, involving neutron diffraction by crystals made of three different isotopes, neutron interferometry, and K_S-meson regeneration.

Quantum theory rests on the superposition principle[1] which asserts that the states of a physical system can be represented as the elements of a linear manifold. That is, if ψ_1 and ψ_2 are two possible states of a system and c_1 and c_2 are arbitrary numbers, then $c_1\psi_1+c_2\psi_2$ is also a possible state of that system. It is usually taken for granted that the coefficients c_1 and c_2 are complex numbers. However, it is possible to imagine a real quantum theory[2] or one based on quaternions.[3-6] In this article, I show how it is possible to distinguish experimentally between real, complex, and quaternion quantum theories.

Real quantum theory, although logically consistent, can be easily ruled out for our world[7]: e.g., complex coefficients are needed in order to combine linearly polarized photons into circularly polarized ones.[8] More generally, correspondence with classical physics leads to the commutation relations $[p,q]=i\hbar$. A formal test, which will later be extended to distinguish between complex and quaternion quantum theories, is the following.

Consider a beam of particles impinging on a scatterer. Let ψ_1 represent the state of the scattered particles, i.e., ψ_1 is the difference between the actual state ψ and the state ψ_0 which we would have if the scatterer were absent. Assume that ψ_0 is normalized to unit flux. Now, set a detector at a distance R from the scatterer and let χ/R represent the state for a unit flux of particles passing through that detector. Then the cross section for scattering into our detector is defined as

$$\sigma_1=|(\chi,\psi_1)|^2,$$

where (χ,ψ_1) denotes the scalar product of the states χ and ψ_1. If this scalar product is a complex number, we can write

$$(\chi,\psi_1)=a_1\exp(i\varphi_1),$$

so that

$$\sigma_1=a_1^2.$$

Similar formulas hold for quaternion quantum theory, with $\exp(i\varphi_1)$ replaced by unimodular quaternion.

Consider now a different scatterer, with scattering amplitude

$$(\chi,\psi_2)=a_2\exp(i\varphi_2).$$

We have likewise

$$\sigma_2=a_2^2.$$

Finally, if both scatterers are present, we have to a good approximation

$$(\chi, \psi_{12}) = (\chi, \psi_1) + (\chi, \psi_2).$$

This relation is valid if double scattering can be neglected. The total cross section thus is

$$\sigma_{12} = |a_1 \exp(i\varphi_1) + a_2 \exp(i\varphi_2)|^2 = \sigma_1 + \sigma_2 + 2(\sigma_1\sigma_2)^{1/2} \cos(\varphi_1 - \varphi_2).$$

Note that σ_{12} as well defined provided that the relative position of the scatterers is held fixed (coherent scattering).

Define

$$\gamma = (\sigma_{12} - \sigma_1 - \sigma_2)/2(\sigma_1\sigma_2)^{1/2}.$$

This quantity involves only cross sections and can therefore be measured for any pair of scatterers. By virtue of the preceding equation, the measurement of γ gives a simple criterion to distinguish between real and complex quantum theories:

If $\gamma = \pm 1$, real quantum theory is admissible. If $|\gamma| < 1$, we may have complex (or quaternion) quantum theory. And if $|\gamma| > 1$, the superposition principle is violated.[9]

At this point, the reader should be warned that the above formulas, which have been derived for pure states, may not be valid for mixtures such as an unpolarized beam, if the cross sections are spin or polarization dependent. For such a mixture, we have *averages*

$$\langle\sigma_{12}\rangle = \langle\sigma_1\rangle + \langle\sigma_2\rangle + 2\langle\sigma_1\sigma_2 \cos(\varphi_1 - \varphi_2)\rangle.$$

In that case

$$\langle\gamma\rangle = (\langle\sigma_{12}\rangle - \langle\sigma_1\rangle - \langle\sigma_2\rangle)/2(\langle\sigma_1\rangle\langle\sigma_2\rangle)^{1/2} = \langle(\sigma_1\sigma_2)^{1/2} \cos(\varphi_1 - \varphi_2)\rangle/(\langle\sigma_1\rangle\langle\sigma_2\rangle)^{1/2}$$

is not the cosine of a phase difference, and the formulas derived in this paper are not valid. (They do remain valid if the cross sections are *not* affected by the spin or polarization variables.)

We now consider a third scatterer and define, as previously, σ_3, σ_{31}, and σ_{23}, and

$$\alpha = (\sigma_{23} - \sigma_2 - \sigma_3)/2(\sigma_2\sigma_3)^{1/2},$$

and

$$\beta = (\sigma_{31} - \sigma_3 - \sigma_1)/2(\sigma_3\sigma_1)^{1/2}.$$

In complex quantum theory, α, β, and γ are the cosines of $(\varphi_1 - \varphi_2)$, $(\varphi_2 - \varphi_3)$, and $(\varphi_3 - \varphi_1)$ and therefore are not independent, since these angles sum up to zero. An elementary calculation gives

$$F(\alpha, \beta, \gamma) \equiv \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma = 1.$$

On the other hand, if the amplitudes (χ, ψ_n) are quaternions, their sums do not behave as vectors in a plane but as vectors in a four-dimensional space. We then have $0 \leq F(\alpha, \beta, \gamma) < 1$. The criterion to distinguish between complex and quaternion quantum theory thus is as follows:

If $F(\alpha, \beta, \gamma) = 1$, complex quantum theory is admissible. If $F < 1$, we may have quaternion quantum theory. And if $F > 1$, the superposition principle is violated.[10,11]

It should now be clear that quaternion quantum theory is essentially different from complex quantum theory. It is not equivalent to having some additional "internal" degree of freedom. Let us devise an experiment to distinguish between the two.

As explained above, the scatterers must act coherently and multiple scattering should be negligible. This rules out some tantalizing ideas, like scattering neutrinos from the three different quarks in baryons.

Apparently, the simplest test is Bragg scattering by crystals made of three different kinds of atoms. Indeed, this test was performed long ago with x rays: The fact that phase angles are coplanar[12] is the basis of the multiple isomorphous replacement method, used to resolve the structure of proteins.[13] However, x-ray diffraction involves only the interaction of photons and electrons, and we should not expect to observe there significant deviations from standard quantum theory.

On the other hand, nuclear forces are not as well understood as quantum electrodynamics. It thus appears that a nontrivial test could be neutron diffraction by crystals made of three different isotopes. The latter should have large neutron *capture* cross sections, for the following reason.

The scattering amplitude can be written as[14]

$$f = [\eta \sin 2\delta + i(1 - \eta \cos 2\delta)]/2k,$$

where k is the wave number, δ is the S-wave phase shift (the other partial waves are completely negligible for thermal neutrons), and η is the elasticity parameter. Both δ and $1 - \eta$ are very small, as can be seen from the scattering and absorption cross sections

$$\sigma_s = \frac{4\pi}{k^2}\left[\eta \sin^2\delta + \left(\frac{1-\eta}{2}\right)^2\right]$$

and

$$\sigma_a = \frac{\pi}{k^2}(1-\eta^2).$$

For thermal neutrons, $4\pi/k^2 \simeq 10^8$ b, i.e., $\delta \simeq 10^{-4}$.

We thus have approximately $f = [\delta + i(1-\eta)/2]/k$, the phase of which will be nontrivial provided that $(1-\eta)/2$ has at least the same order of magnitude as δ. This implies that σ_a should be of the order of 10^4 b or more, for at least two of the scatterers. Usually σ_a is much smaller, and f almost real. If this happens for scatterers 1 and 2, say, then $\alpha \simeq \beta$ and $\gamma \simeq 1$, so that $F(\alpha,\beta,\gamma) \simeq 1$ trivially. Thus, most such experiments cannot distinguish between complex and quaternion quantum theories.

There are, however, several stable nuclides with capture cross sections of the required magnitude. The two largest ones are ${}_{64}\mathrm{Gd}^{157}$ and ${}_{64}\mathrm{Gd}^{155}$, but Gd is ferromagnetic and this might complicate the data analysis. Next, we have ${}_{62}\mathrm{Sm}^{149}$ (41 000 b, $I=\frac{7}{2}$) and ${}_{48}\mathrm{Cd}^{113}$ (20 000 b, $I=\frac{1}{2}$) or, if even-even nuclei are preferred in order to avoid the incoherent background due to nuclear spin,[15] one could use ${}_{66}\mathrm{Dy}^{164}$ (3700 b) and ${}_{70}\mathrm{Yb}^{168}$ (3200 b). The third isotope need not have a large capture cross section, i.e., its f may be almost real, since only relative phases are important.

Instead of Bragg scattering, another possibility could be neutron interferometry.[16] As the latter involves only the forward-scattering amplitude, this test would have less generality than the one discussed above, but it might be easier to perform. Consider a plane wave $e^{i\vec{k}\cdot\vec{r}}$. Passage through a thin plate changes the amplitude into $Te^{i\Delta}e^{i\vec{k}\cdot\vec{r}}$. The transmission coefficient T is due to absorption in the plate and reflections at its surfaces. The phase shift Δ is due to the difference in optical path. By means of interference with a reference beam $e^{i\vec{k}'\cdot\vec{r}}$, with $\vec{k}' \simeq \vec{k}$, it is possible to measure both T and Δ. Now consider two plates made of different materials, taken singly and jointly. The total transmission coefficient T_{12} will not, in general, be T_1T_2 because of multiple reflections between the plates. But the total phase shift Δ_{12} should be $\Delta_1+\Delta_2$, if our use of complex numbers is legitimate.

On the other hand, quaternion interference will usually give $\Delta_{12} \neq \Delta_1 + \Delta_2$, because quaternion rotations do not commute. To test this with the strongly absorbing materials mentioned above, very thin lamellae (a few microns) should be used, yet they should not be so thin as to make the Δ too small.

Still another possible test could be comparing K_S-meson regeneration[17] by three different materials, taken singly[18] and pairwise. Here, the observed quantity is the square of the forward regeneration amplitude. For our purpose, it is similar to a cross section and the expression $F(\alpha,\beta,\gamma)$ can be defined exactly as before. For this test, it would be especially interesting to combine neutron-rich and proton-rich nuclei.

In summary, I have provided a simple quantitative distinction between real, complex, and quaternion quantum theories, based on experimental tests of a universal nature, independent of our knowledge—or ignorance—of the dynamical laws.

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[1]P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford Univ. Press, Oxford, 1947), pp. 16–18.

[2]E. C. G. Stueckelberg *et al.*, Helv. Phys. Acta 33, 727 (1960), and 34, 621, 675 (1961), and 35, 673 (1962).

[3]Quaternions are hypercomplex numbers which can be written as $a+ib+jc+kd$, where $i^2=j^2=k^2=-1$ and $ij=-ji=k$, etc. They are the only generalization of complex numbers satisfying the associative and distribution laws and for which division is possible and is unique. See C. Chevalley, *Theory of Lie Groups* (Princeton Univ. Press, Princeton, N. J., 1946), pp. 16–18.

[4]T. Kaneno, Prog. Theor. Phys. 23, 17 (1960).

[5]D. Finkelstein *et al.*, J. Math. Phys. (N.Y.) 3, 207 (1962), and 4, 136, 788 (1963).

[6]G. Emch, Helv. Phys. Acta 36, 739, 770 (1963).

[7]Stueckelberg's "real" quantum theory (see Ref. 2) requires the introduction of an operator J satisfying $J^2=-1$ and commuting with all observables. As a consequence, the states ψ and $J\psi$ are linearly independent, although they are physically indistinguishable. And the definition of the scalar product involves i explicitly; see Eq. (*A*-2.7), p. 747.

[8]Polarized light also exists classically. However, in classical physics $\vec{E}$ and $\vec{B}$ can be measured exactly at each point of space time. In particular, for a plane wave $\vec{E}=\vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t)$, both the amplitude $\vec{E}_0$ and the phase $\vec{k}\cdot\vec{r}-\omega t$ are well defined. On the ohter hand, for a single photon, or any definite number of photons, the phase of the field is completely undetermined, since it is the variable canonically conjugate to the number operator.

[9]Abandoning the superposition principle was suggested by B. Laurent and M. Roos, Phys. Lett. 13, 269 (1964), and Nuovo Cimento 40A, 788 (1965).

[10]F can never be negative if $\alpha^2+\beta^2+\gamma^2 \leq 3$.

[11]No new information is obtained by considering the three scatterers simultaneously, because $\sigma_{123}=\sigma_{12}+\sigma_{23}+\sigma_{31}-\sigma_1-\sigma_2-\sigma_3$ for all types of quantum theory.

[12]D. Harker, Acta Cryst. 9, 1 (1956).

[13]T. L. Blundell and L. N. Johnson, *Protein Crystallography* (Academic, New York, 1976), p. 161.

[14]S. Gasiorowicz, *Quantum Physics* (Wiley, New York, 1974), p. 384.

[15]W. Marshall and S. W. Lovesey, *Theory of Thermal Neutron Scattering* (Oxford Univ. Press, Oxford, 1971), p. 11. Note that we must already cope with an incoherent background due to the presence of two isotopes. These must either be completely miscible in a single crystal lattice, or form stoichiometric compounds, because it is essential to achieve good homogeneity (chemical, isotopic, and in density).

[16]U. Bonse and W. Graeff, in *X-Ray Optics*, edited by H.-J. Queisser (Springer-Verlag, Berlin, 1977).

[17]K. Kleinknecht, Annu. Rev. Nucl. Sci. 26, 1 (1976).

[18]A. Gsponer *et al.*, Phys. Rev. Lett. 42, 13 (1979).