Turbulent Destabilization and Saturation of the Universal Drift Mode in a Sheared Magnetic Field

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In a sheared magnetic field, turbulent diffusion of electrons in the vicinity of a mode rational surface can eliminate the stabilizing influence of nonresonant electrons and lead to an absolute instability at small but nonzero wave amplitudes. As the turbulence grows, the inverse electron Landau resonance is broadened in both velocity and configuration space, and the convective shear damping due to ions is enhanced by turbulent spatial broadening of the mode until saturation occurs.

The original work of Pearlstein and Berk[1] indicated the existence of an absolute universal instability of a confined plasma ($\nabla p \neq 0$) in a sheared magnetic field. Recently, numerical integration of the exact differential equation describing the radial structure of the drift-wave eigenmode showed the absence of an absolute instability, regardless of how weak the shear or how large the poloidal wave number.[2,3] The stability of the universal mode in these improved treatments is due to the inclusion of nonresonant electrons in the region about the mode rational surface where $k_\parallel(r) = [m - nq(r)]/Rq \lesssim \omega/v_{\mathrm{th}e}$. Here, m and n are poloidal and toroidal mode numbers, respectively, $q(r) = rB_T/RB_p$ is the safety factor, ω is the mode frequency, and $v_{\mathrm{th}e} = (2T_e/m_e)^{1/2}$ is the electron thermal speed. Physically, for $(k_\parallel v_{\mathrm{th}e}/\omega)^2 < 1$, the local electron response is (linearly) nonadiabatic and therefore does not support the drift-wave oscillation imposed by the global mode. Nonresonant damping results. Thus, instability might be recovered by altering the electron response in the region around the rational surface.

In this Letter it is shown that turbulent diffusion of electrons across the rational surface, due to a combination of shear ($\partial k_\parallel/\partial r \equiv k_\parallel' \neq 0$) and random $\vec{E}\times\vec{B}$ fluctuations and/or stochastic magnetic perturbations, results in a finite-amplitude-induced version of the absolute universal instability. Physically, the turbulent scattering of electrons across the rational layer leads to an effective finite value for $k_\parallel$ which destroys the stabilizing influence of the nonresonant electrons. To estimate the level of turbulence required for the onset of nonlinear instability, we note that in a wave period an electron will diffuse radially a distance $\Delta x = (D_{rr}\omega^{-1})^{1/2}$, where D_{rr} is the radial turbulent diffusion coefficient. Therefore, $(k_\parallel)^{\mathrm{eff}} \equiv k_\parallel'\Delta x \gtrsim \omega/v_{\mathrm{th}e}$ implies the threshold $[D_{rr}(k_\parallel' v_{\mathrm{th}e})^2/3]^{1/3} \equiv \omega_c \gtrsim \omega$. Thus, a diffusion coefficient only somewhat larger than the classical value $D_c \sim \rho_e^2\nu_{ei}$ is sufficient to destabilize the universal mode, provided shear damping can be overcome. For electrostatic turbulence, this requires $\tilde{n}/n \sim 10^{-4}$. At larger amplitudes, the electron growth is reduced and the ion shear damping is enhanced by spatial broadening of the mode, yielding nonlinear stabilization.

The turbulent diffusion process in a sheared magnetic field produces a resonance broadening mechanism for the electrons which is fundamentally different from the process, due to random $\vec{E}\times\vec{B}$ drifts alone, in a shearless field.[4,5] With shear, stochastic radial motion combines with parallel electron streaming to induce random poloidal motion. The decorrelation frequency resulting from this random motion of electrons in a sheared field can be estimated as follows. In a correlation time τ_c, electrons diffuse radially a distance $L_\perp = (D_{rr}\tau_c)^{1/2}$ with a change $\Delta k_\parallel = k_\parallel' L_\perp$. In the same time, they free stream *along* the magnetic field a distance $L_\parallel = v_\parallel\tau_c$. By definition, the phase change in a correlation time is one wavelength, $L_\parallel\Delta k_\parallel = 1$, which yields $\tau_c = \omega_c^{-1}$. Comparing τ_c with the correlation time $\tau_{c0} = (k_r^2 D_{rr})^{-1}$ in a uniform field, note that

$$\xi \equiv \tau_c/\tau_{c0} \approx [L_s D_{rr}(k_\theta \Delta r^3 v_{\mathrm{th}e})^{-1}]^{2/3},$$

where $L_s = Rq^2/(rq')$ and $\Delta r \gtrsim k_\theta^{-1}$ is the radial mode width. For tokamaks, $\xi \ll 1$, except near the value of D_{rr} required for saturation of short-wavelength modes, where $\xi \lesssim 1$. Thus, the shear decorrelation gives the dominant nonlinear broadening in a tokamak plasma.

The electron distribution function for a turbulent plasma in a sheared magnetic field is written $f_e = \bar{F}_e + \tilde{f}_e$, where $\bar{F}_e$ is the phase-averaged part of f_e and $\tilde{f}_e$ is the fluctuating response. With $\vec{E} = -\nabla\Phi$, $\epsilon = \frac{1}{2}m_e v^2$, and $\vec{n} = \vec{B}/B$, the fluctuating part of f_e satisfies the nonlinear drift equation

$$\left[\frac{\partial}{\partial t} + v_\parallel \vec{n}\cdot\nabla - \frac{\nabla\Phi\times\vec{B}}{B^2}\cdot\frac{\partial}{\partial\vec{r}} + |e|v_\parallel \vec{n}\cdot\nabla\Phi\frac{\partial}{\partial\epsilon}\right]\tilde{f}_e = (-|e|v_\parallel\vec{n}\cdot\nabla\Phi + i\omega_{*e}|e|\Phi)\partial\bar{F}_e/\partial\epsilon. \quad (1)$$

Here

$$\omega_{*e} \equiv [i\vec{n}\times\nabla\bar{F}_e/(|e|B\partial\bar{F}_e/\partial\epsilon)]\cdot\nabla\ln\Phi = k_\theta(\partial\bar{F}_e/\partial r)(|e|B\partial\bar{F}_e/\partial\epsilon)^{-1}$$

is the electron diamagnetic frequency and $k_\theta = m/r$ is the poloidal wave number. The effects of magnetic field fluctuations have been neglected here, although for $\beta_e > m_e/m_i$ they also scatter the electron orbits.[6]

Integrating Eq. (1) along perturbed electron trajectories yields for the phase-coherent part[4] of $\tilde{f}_e = \sum \tilde{f}_e^{\vec{k}}(r)\exp[-i\omega t + i(m\theta - n\varphi)]$

$$\tilde{f}_e^{\vec{k}} = -|e|\Phi_{\vec{k}}\partial\bar{F}_e/\partial\epsilon - i(\omega - \omega_{*e})|e|(\partial\bar{F}_e/\partial\epsilon)R_{\vec{k}}\Phi_{\vec{k}}, \quad (2)$$

where $\Phi = \sum\Phi_{\vec{k}}(r)\exp[-i\omega t + i(m\theta - n\varphi)]$ and the resonance operator is[7] (in the cumulant approximation)

$$R_{\vec{k}}\Phi_{\vec{k}} = \int_0^\infty \exp[i\omega' t' - \tfrac{1}{2}m^2\langle\delta\theta^2\rangle - \tfrac{1}{2}(k_\parallel' v_\parallel)^2\langle(\int_0^{t'}\delta r\, dt'')^2\rangle]G(\Phi_{\vec{k}})\,dt'. \quad (3)$$

Here, $\omega' = \omega - k_\parallel v_\parallel$ and

$$G(\Phi) = \int_{-\infty}^{\infty}\frac{dr'}{[2\pi\langle\delta r^2(t')\rangle]^{1/2}}\exp\{-(r-r')^2/[2\langle\delta r^2(t')\rangle]\}\Phi(r'). \quad (4)$$

The terms $\propto\langle\delta\theta^2\rangle$ and $\langle\delta r^2\rangle$ have been previously computed for a uniform magnetic field.[4,5] The new broadening term $\propto(k_\parallel' v_\parallel)^2$ represents an additional stochastic change in θ, due to nonlinear motion, $d\delta\theta/dt' = -(v_\parallel/Rq)(\partial\ln q/\partial r)\delta r(t')$, which gives the dominant broadening ($\propto\omega_c$) in a sheared field.

The average displacements appearing in Eqs. (3) and (4) may be evaluated by substituting $\tilde{f}_e^{\vec{k}}$ from Eq. (2) into the quasilinear equation[4,6] for $\bar{F}_e$, yielding $\langle\delta\theta^2\rangle = 2D_{\theta\theta}t'/r^2$ and $\langle\delta r^2\rangle = 2D_{rr}t'$. Here, the diffusion tensor for electrostatic turbulence is

$$\overleftrightarrow{D} = (c/B)^2\sum_{\vec{k}}\langle\vec{n}\times\nabla L_{-\vec{k}}\vec{n}\times\nabla R_{\vec{k}}L_{\vec{k}}\rangle, \quad (5)$$

with $L = \Phi - (v_\parallel/c)A_\parallel$. Noting that $\langle\delta r(t_1)\delta r(t_2)\rangle = \langle\delta r^2(|t_1 - t_2|)\rangle = 2D_{rr}|t_1 - t_2|$, Eq. (3) becomes

$$R_{\vec{k}}\Phi_{\vec{k}} = \int_0^\infty d\tau\,\exp[i\omega' t' - t'/t_{c\theta} - (t'/t_c)^3]G(\Phi), \quad (6)$$

where $t_{c\theta}^{-1} = k_\theta^2 D_{\theta\theta}$ is the shearless decorrelation frequency,[4] and $t_c^{-1} = [(k_\parallel' v_\parallel)^2 D_{rr}/3]^{1/3}$ is the decorrelation frequency in a sheared magnetic field, which vanishes in the absence of wave-particle energy transfer. As noted previously, for tokamaks $\xi = (\omega_c t_{c\theta})^{-1} < 1$, and thus $G(\Phi) \approx \Phi(r) + D_{rr}t'\partial^2\Phi/\partial r^2$, representing turbulent broadening of Φ over a correlation length $L_c = (D_{rr}t_c)^{1/2}$. This contrasts with the shearless case where $\Delta r^2/L_c^2\xi^{-1} \sim 1$ and $G(\Phi) = \Phi(r)\exp(-k_r^2 D_{rr}t')$ contributes to the resonant wave-particle energy transfer.

Using Eq. (6) to calculate the electron density perturbation, assuming $\bar{F}_e$ is a Maxwellian, and invoking the linear ion response (for $\xi < 1$) together with quasineutrality yields the eigenmode equation:

$$\partial^2\Phi_{\vec{k}}/\partial x^2 - [\Lambda - \mu^2 x^2 + \sigma(x)/x]\Phi_{\vec{k}} = 0. \quad (7)$$

In Eq. (7), $x = (r - r_0)/\rho_i$, where $\rho_i = (T_i m_i)^{1/2}/eB$ is the ion Larmor radius, $q(r_0) = m/n$ defines the location r_0 of the rational surface, and $\Lambda = [1 + \tau(1 - \Gamma_0) - \Gamma_0\omega_{*e}/\omega]d^{-1}$ contains the basic drift-wave response. Here, $\tau = T_e/T_i$, $\Gamma_n = I_n(b)\exp(-b)$, and $b = (k_\theta\rho_i)^2$. The shear parameter is

$$\mu = \tau^{-1}(L_n/L_S)(\omega_{*e}/\omega)[\Gamma_0(\tau + \omega_{*e}/\omega)d^{-1}]^{1/2},$$

with $L_n^{-1} = -\partial\ln n/\partial r$. The destabilizing electron contributions are contained in $\sigma(x) = \sigma_0 Z[(x_e + ix_c)/x]$, where $\sigma_0 = (\omega/\omega_{*e} - 1)\alpha d^{-1}$, $x_e = \alpha\omega/\omega_{*e}$, $x_c = \alpha\omega_c/\omega_{*e}$, $\alpha = (\frac{1}{2}\tau m_e/m_i)^{1/2}L_S/L_n$, and Z is the plasma dispersion function. A Lorentzian form for the resonance function was chosen for analytic and numerical convenience, although the results given here are not sensitive to this approximation. The quantity

$$d = (\Gamma_0 - \Gamma_1)(\tau + \omega_{*e}/\omega) + 3i(1 - \omega/\omega_{*e})(\omega_{*e}/\omega_c)x_c^2$$

includes both ion gyroradius and turbulent broadening effects.

The turbulence enters the electron response function $\sigma(x)$ through the effective collision frequency ω_c. However, turbulence does not affect the electrons in the same way as a local (in real space) number-conserving collision operator, which is known to have a stabilizing influence on drift waves in slab geometry.[8] Indeed, the $\vec{E}\times\vec{B}$ (or magnetic) fluctuations scatter the particle orbits in real space, producing a turbulent flux of electrons in the kinetic Eq. (1) for f_e.

For $\omega_c \ll \omega$, Eq. (7) reduces to the eigenvalue problem solved in Refs. 2 and 3. As indicated previously, small turbulence levels can produce $\omega_c \gtrsim \omega$. Thus, the effects of turbulence are well illustrated in the limit $\omega/\omega_c < 1$. Then the Z function in the electron response becomes purely imaginary and there is no longer any nonresonant electron contribution, which previously led to stabilization of the linear universal mode in a quiescent plasma.[2,3]

The destabilizing electron contribution to Eq. (7) can be treated by perturbation theory (particularly for $\omega_c \gtrsim \omega$ where the numerically determined eigenfunctions do not depart significantly from the Weber functions[1]) using the *full* electron Z function. The dispersion relation for the most unstable modes becomes

$$\Lambda + i\mu + \Sigma_e(x_e, x_c) = 0, \tag{8}$$

where

$$\Sigma_e = \int_0^\infty \varphi_0^2(x)[\sigma(x)/x]\,dx \Big/ \int_0^\infty \varphi_0^2\,dx,$$

and

$$\varphi_0(x) = \exp(-i\mu x^2/2)$$

is the lowest-order eigenmode corresponding to the propagation of energy away from the rational surface[1] for $x > |\sqrt{\mu}|$. Treating $\partial\omega_c/\partial x \approx 0$ for $|x| < |\sqrt{\mu}|$ yields

$$\Sigma_e = 2i\sigma_0(i\mu)^{1/2}H(-2i(x_e + ix_c)(i\mu)^{1/2}),$$

where $H(z) = \int_0^\infty e^{-zt}(1+t^2)^{-1/2}dt$. The branch $\mathrm{Re}(i\mu)^{1/2} > 0$ for $\mathrm{Im}\mu < 0$ is chosen.

For values of $\omega_c/\omega_{*e} \gtrsim 0.1$ (corresponding to $\omega_c/\mathrm{Re}\omega \gtrsim 1$), the figure shows that with $k_\theta\rho_i \gtrsim 1$ and moderate shear ($L_S/L_n = 16$), the turbulence destabilizes the drift mode, with maximum growth rates $\mathrm{Im}\omega/\mathrm{Re}\omega \sim 0.2$. There is good agreement between the numerical results using the shooting code described in Ref. 2 (which predicted stability for $\omega_c = 0$) and the analytic dispersion relation Eq. (8). (See Fig. 1.) As the turbulence increases, the electron growth arising from Σ_e is weakened and finally reduced to a value where shear damping, enhanced by turbulent spatial broadening, leads to stabilization. There is a narrow range of values for ω_c, corresponding to a variation in D_{rr} over three orders of magnitude, over which the nonlinear instability is excited and finally saturates.

The value of ω_c, and hence the turbulent diffusion coefficient, required for saturation of this instability can be determined by solving Eq. (8) at marginal stability. As the turbulence grows, $x_c|\sqrt{\mu}|$ approaches unity (the mode width is limited to $\Delta x \gtrsim x_c$ by turbulent broadening). In this limit, the approximate stability criterion becomes (for $b \gtrsim 1$, corresponding to the modes most difficult to stabilize at fixed shear)

$$(\omega_{*e}/\omega_c)^2 - \mu_0^2(\omega_{*e}/\omega)(8\pi b^3)^{-1/2} - (1.5\mu_0^2 x_c^2)^2 = 0, \tag{9}$$

where $\Lambda_0 = \Lambda d$ and $\mu_0 = \mu d^{1/2}$. In Eq. (9), the first term represents broadened electron growth, the second is due to linear shear damping, and the last is enhanced shear damping resulting from the turbulent spatial broadening of the mode, which decreases the effective shear length [the Φ'' term in $G(\Phi)$]. A stabilization mechanism similar to this latter one has been computed for a Q machine in Ref. 9. There, however, $\tau_c/\tau_{c0} \gg 1$, so that mode coupling in the *ion* kinetic

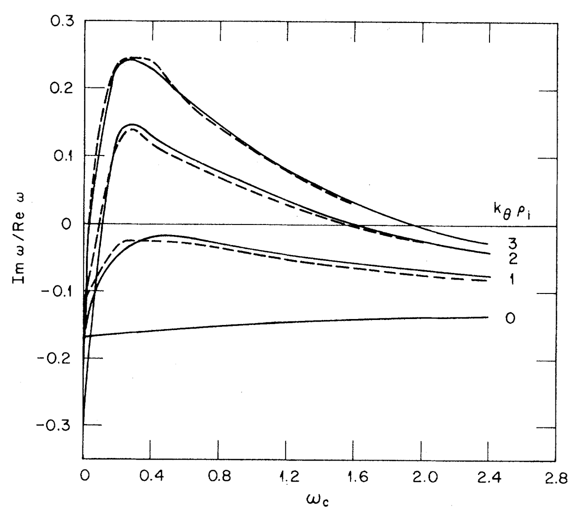

FIG. 1. Growth rate (normalized to real frequency) vs ω_c (normalized to ω_{*e}) for $T_e/T_i = 1$, $L_S/L_n = 16$, and various values of $k_\theta\rho_i$, obtained numerically (solid line) and from analytic dispersion relation (dashed line).

equation was the dominant nonlinearity (with adiabatic electrons).

The maximum diffusion coefficient obtained from Eq. (9) occurs for $b = b_0 \equiv (1+\tau)^3\tau^{-2}\Delta_{PB}^{-1}$, where the nonlinear and linear shear damping become comparable. Here, $\Delta_{PB} \equiv (L_S/L_n)^3(m_e/m_i)$ is of order unity for tokamaks. The mode width $\Delta x \sim (\tau^3\Delta_{PB}L_S/L_n)^{1/2}(1+\tau)^{-2} \gtrsim 1$, which justifies the use of the differential Eq. (7). The diffusion coefficient which results from maximizing D_{rr} with respect to b is

$$D_{rr} = 15\Delta_{PB}^{3/2}\tau^{5/2}[0.5(1+\tau)]^{-9/2}\rho_S^2 c_S/L_S, \qquad (10)$$

where $c_S = (T_e/m_i)^{1/2}$ and $\rho_S^2 = \tau\rho_i^2$. The associated electron thermal conduction coefficient is $\kappa_e \approx \frac{3}{2}D_{rr}$. If $\vec{E}\times\vec{B}$ electrostatic turbulence is the dominant scattering mechanism for these modes, then Eqs. (5) and (10) indicate a density fluctuation level at saturation, $\tilde{n}/n = \Delta_{PB}(\rho_s/L_n)$ for $\tau = 1$, in the strong-turbulence limit $\omega_c \gtrsim \omega'$. The coefficient in Eq. (10) is of the correct order of magnitude to account for electron heat transport in tokamaks outside the $q = 1$ surface, with a fluctuation level of several percent.

In conclusion, destabilization and saturation of the drift mode in a sheared field have been shown to result from a resonance broadening mechanism that dominantly affects electrons. This contrasts with previous turbulence theories in a shearless field,[4] where nonlinear ion damping led to saturation and the electron dynamics were linear. The present theory predicts saturation at modest fluctuation levels.

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Maximum Energy-Confinement Time in Joule-Heated Tokamaks

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Measurements of the energy confinement time τ_E in the ISX-A (Impurity Study Experiment) tokamak are interpreted theoretically using a one-dimensional time-dependent transport code. The maximum τ_E observed as the plasma density is varied over a wide range occurs at that density above which anomalous electron thermal conductivity leads to a smaller energy flux than neoclassical ion thermal conductivity.

One of the most striking features of recent experiments in the ISX-A (Impurity Study Experiment) tokamak[1] is the apparent saturation of the energy confinement time τ_E with increasing plasma density as shown in Fig. 1. Effective impurity control in the ISX-A, as in the earlier Alcator experiments,[2] permitted operation over a comparatively wide range of plasma density under circumstances such that radiation was not a dominant energy-loss mechanism in the interior of the plasma. Since anomalous heat losses decrease with density, while neoclassical losses

