is only six, obtained from the maximally symmetric ground states.

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Superfluid Two-Phase Sound

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The properties of the sound wave propagating in a superfluid two-phase system (liquid helium and its vapor) are investigated. At a frequency sufficiently low so that the superfluid helium and its vapor are in quasistatic equilibrium, there can propagate only one sound mode whose wave-guide–like velocity depends strongly on the vapor properties as well as the velocity of second sound. Near the λ point, the mode probes the critical properties of the specific heat of the superfluid.

The strong coupling between second sound in superfluid helium and the ordinary sound in the vapor above it gives rise to an unusual type of sound which propagates in the two-phase system. The fact that the entropy convects in a superfluid at a velocity different from the velocity of the center of mass is basic to the existence of second sound, a propagating thermal wave. At an interface between a superfluid and its vapor, this internal convection modifies the boundary condition that follows from conservation of mass and entropy at the interface. This new boundary condition [Eq. (15) below] leads to the new two-phase sound mode. Whereas the speed of sound for a classical fluid in such an arrangement is an interpolation between the velocities of sound in the vapor and liquid (in general weighted heavily toward the vapor), the superfluid two-phase sound involves new independent thermodynamic quantities, in particular the entropy of the vapor. Experimental observations confirm many aspects of the theory.

The position ξ of a liquid-vapor interface can move due to a convection of fluid as well as evaporation and condensation. From basic conservation laws[1] one then obtains

$$(\rho-\rho_v)\dot{\xi}=(J-J_v)_\perp, \quad (1)$$

$$(\rho s-\rho_v s_v)\dot{\xi}=(f-f_v)_\perp, \quad (2)$$

$$(\rho v_i-\rho_v v_{vi})\dot{\xi}=P_{i\perp}-P_{vi\perp}, \quad (3)$$

where ρ, s, and $\vec{v}$ are the mass density, specific entropy, and velocity field; $\vec{J}$, $\vec{f}$, and P_{ij} are the mass, entropy, and momentum flux densities; the subscript v denotes a vapor variable and the subscript $\perp$ indicates the component perpendicular to the surface. For the linear theory at issue here the entropy law (2) is a valid conservation law.

For a classical fluid in the Euler approximation, Eqs. (1), (2), and (3) yield, through eliminating $\dot{\xi}$, two boundary conditions for the fluid variables at the free surface:

$$p = p_v, \tag{4}$$

$$v_\perp = v_{v\perp}, \tag{5}$$

where p is the pressure. We consider sound propagation in the z direction parallel to the free surface (see Fig. 1) and choose velocity potentials whose perpendicular derivatives vanish at the solid boundaries:

$$\varphi_v = A\cos[k_v(H-x)]e^{i(kz-\omega t)}, \tag{6}$$

$$\varphi = B\cos(k_1 x)e^{i(kz-\omega t)}. \tag{7}$$

The x components of the wave vectors, k_1 and k_v, are determined by requiring that φ_v and φ satisfy the wave equation, giving

$$\omega^2 = c_v^{\,2}(k^2 + k_v^{\,2}), \tag{8}$$

$$\omega^2 = c_1^{\,2}(k^2 + k_1^{\,2}), \tag{9}$$

where c_v and c_1 are the velocities of sound in the vapor and in the liquid. Applying the boundary conditions (4) and (5) using the definitions $\vec{v} = \nabla\varphi$ and $p = -\rho\,\partial\varphi/\partial t$ yields for the speed ($c = \omega/k$) of the traveling mode at long wavelength ($kH \ll 1$)

$$c^2 \simeq c_1^{\,2}\left(1 + \frac{c_v^{\,2}/c_1^{\,2} - 1}{1 + (c_v^{\,2}/c_1^{\,2})(\rho_v/\rho)[L/(H-L)]}\right). \tag{10}$$

For the common case $\rho_v/\rho \ll 1$ this reduces to $c \simeq c_v$, and for the classical fluid there is little coupling between liquid and vapor.

In the superfluid case Eqs. (1), (2), and (3) also yield boundary conditions which are, however, not enough since now on the reversible level there are three modes to match at the free surface, the extra mode being second sound. As the energy-conservation law for a superfluid is independent from those for mass, entropy, and momentum, one might expect it to yield the necessary condition. This is not the case because on

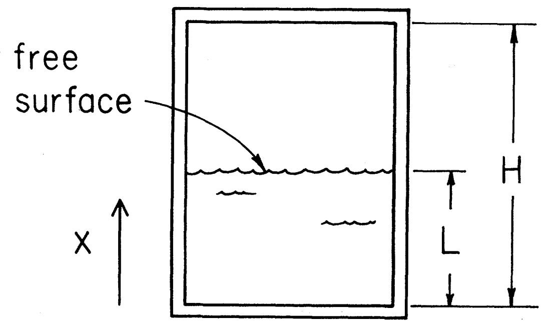

FIG. 1. A schematic cross section of the wave-guide geometry employed. The direction of propagation of the sound mode is the z axis out of the page.

the linear level the energy boundary condition is redundant.

Variables such as temperature which describe reversible effects in the He II correspond with irreversible phenomena (e.g., thermal conduction) in the vapor. Thus one must introduce the dynamical coefficients describing the evaporation-dondensation exchange with the vapor. Following Bergman's approach for third sound[2] we set at the free surface

$$(\rho\dot{\xi} - \rho v_\perp) \equiv \dot{M} = -K(T - T_v) - K_\mu(\mu - \mu_v), \tag{11}$$

$$(\rho s\dot{\xi} - f_\perp) \equiv \dot{S} = -K_T(T - T_v) - K(\mu - \mu_v), \tag{12}$$

where K, K_μ, and K_T are transport coefficients describing the off-equilibrium mass (entropy) flows which result from chemical-potential (μ) and temperature (T) differences, and where $\dot{M}$ ($\dot{S}$) is the mass (entropy) per unit area of liquid to evaporate. Equations (1), (2), (3), (11), and (12) yield four boundary conditions for the fluid variables connecting the four modes (including thermal diffusion in the vapor) which interact at the interface. For He II we set

$$\vec{f} = \rho s\vec{v}_n, \tag{13}$$

whereas for the vapor

$$\vec{f}_v = \rho_v s_v\vec{v}_v - (\kappa_v/T)\nabla T_v. \tag{14}$$

For a superfluid interacting with its vapor we obtain instead of Eq. (5) the new boundary condition

$$\rho s(v - v_n)_\perp = \rho_v s_v(v - v_v)_\perp + \rho_v s(v_v - v_n)_\perp + (\kappa_v/T)[(\rho - \rho_v)/\rho]\nabla_\perp T_v, \tag{15}$$

whereas Eq. (4) remains unchanged. Solving (1) for $\dot{\xi}$ and substituting into (11) and (12) yields the remaining conditions sufficient to match up the four modes.

We look for a solution in the same geometry as Fig. 1 with velocity potential and temperature in the vapor given by

$$\varphi_v = A\cos[k_v(H-x)]e^{i(kz-\omega t)}, \quad (16)$$

$$\delta T_v = \{(\partial T/\partial P)_s i\omega\rho_v A\cos[k_v(H-x)] + C\cos[k_{Dv}(H-x)]\}e^{i(kz-\omega t)}, \quad (17)$$

where k_{Dv} is determined by the thermal diffusion equation, giving

$$i\omega = \chi_v(k^2 + k_{Dv}^2) \quad (18)$$

with χ_v the ratio of thermal conductivity to specific heat per unit volume. In the liquid the velocity potentials for the center-of-mass and normal-fluid second-sound motion are

$$\varphi = B\cos(k_1 x)e^{i(kz-\omega t)}, \quad (19)$$

$$\varphi_{n2} = D\cos(k_2 x)e^{i(kz-\omega t)}, \quad (20)$$

where

$$\omega^2 = c_2^2(k^2 + k_2^2) \quad (21)$$

with

$$c_2^2 = (\rho_s/\rho_n)[s^2/(\partial s/\partial T)_p] \quad (22)$$

being the velocity of second sound.

Substituting these relations into the boundary conditions yields a dispersion law for the speed and attenuation of the mode. At low frequency the problem can be greatly simplified by noting that the left-hand side of (11) and (12) is higher order in the frequency than the right-hand side. Thus at sufficiently low frequency the temperature and chemical potentials must be equal. In this way the thermal conductivity can also be neglected and the boundary conditions

$$p = p_v, \quad (23)$$

$$\rho s(v - v_n)_\perp \cong \rho_v s_v(v - v_v)_\perp, \quad (24)$$

$$\delta p/\delta T = dp/dT \cong \rho_v s_v \quad (25)$$

(where dp/dT is the derivative along the coexistence curve) are sufficient to match the three propagating modes.

Neglecting ρ_v/ρ and $(\partial\rho/\partial T)_p$ the resulting velocity of propagation is

$$c^2 = c_2^2\left(1 + \frac{c_v^2/c_2^2 - 1}{1 + (c_v^2/c_2^2)[L/(H-L)][\rho s^2\rho_s/(\rho_n s_v\, dp/dT)]}\right). \quad (26)$$

There is only mode possible in the arrangement considered, and its velocity is independent of the bulk modulus of the fluid. The velocity interpolates between the two limits c_v and c_2 as the liquid level L increases. It is important to note that (26) is the low-frequency result. At high ω Eqs. (11) and (12) would lead to the classical result (10). At low frequencies, surface tension can be neglected and furthermore gravity is a negligible force for the problem at hand.

Preliminary measurements showing the existence of this sound mode are displayed in Fig. 2, where the velocity is plotted as a function of the fractional filling L/H. The velocities are measured by observing the resonant frequencies of an annular channel 1.1 cm wide, 1.0 cm deep, and 11.4 cm in mean circumference. The fundamental frequencies range between 150 and 700 Hz. The waves are generated with a heater wire (using ~0.5-mW dissipation) and detected with resistance thermometers (Allen-Bradley nominal 200-Ω, $\frac{1}{8}$-W resistors) in either the liquid or the vapor. The cell is gradually filled by condensing in measured amounts of helium gas. The data points in Fig. 2 are taken with the temperature held constant at 1.26 K. The solid line is Eq. (26) evaluated at this temperature, and the agreement

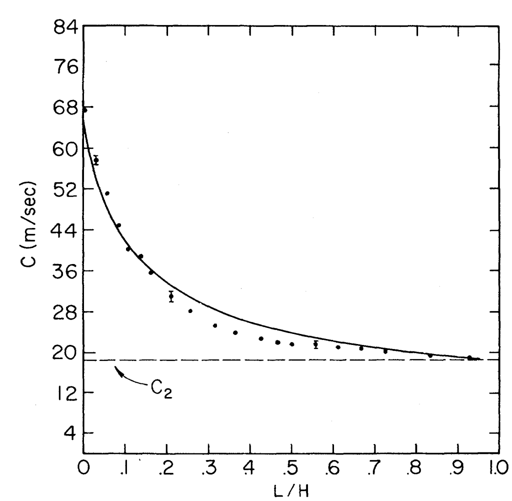

FIG. 2. Velocity of the low-frequency two-phase sound mode at 1.26 K as a function of liquid level. The dashed line indicates the velocity of second sound, and the solid line is the theory, Eq. (26). The vapor velocity at this temperature is 66 m/sec.

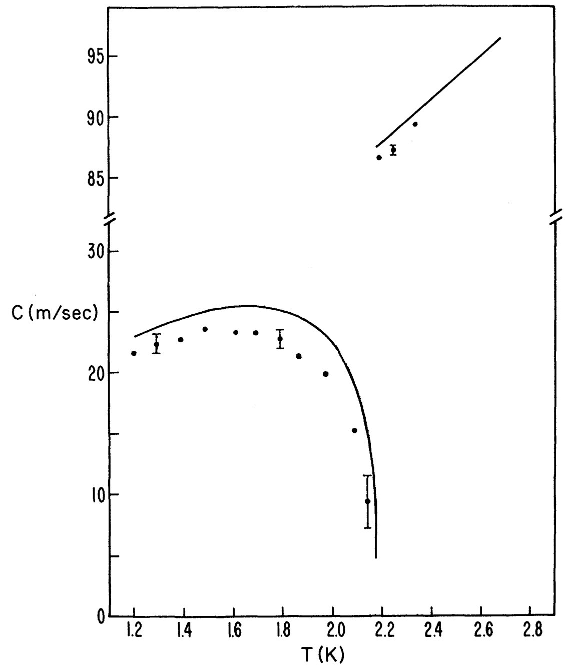

FIG. 3. Temperature dependence of the mode at $L/H = 0.55$. The solid lines are the theory.

is good, although in the region $0.2 < L/H < 0.6$ the data points lie somewhat below the theoretical curve. At this point we do not know whether the discrepancy arises from experimental effects, or is due to the approximations inherent in Eq. (26) where dissipation is neglected. The quality factor of the mode is generally quite low, starting from a Q of 27 at $L/H = 0$ and then dropping rapidly to a $Q \sim 5$ at $L/H = 0.2$. The Q then gradually increases for values of L/H greater than 0.5, reaching a value of ~ 20 just before the channel fills completely.

Figure 3 shows the temperature dependence of the mode at a fixed liquid level $L/H = 0.55$. There are several interesting features near the λ point, 2.17 K. As the λ transition is approached from below, the dominant contribution to the sound velocity of Eq. (26) is given by

$$c^2 = [(H - L)/L](\rho_v/\rho)[s_v^2/(\partial s/\partial T)_p] \qquad (27)$$

which will go to zero at the transition if and only if the specific heat displays an infinity. For a value of the specific heat of $32R$ suggested by Ahlers[3] the phase velocity for $L = H/2$ is about 5 m/sec, which is much higher than c_2 but much smaller than c_v. Just above T_λ there is a discontinuous jump in the velocity to the value c_v. This is observed experimentally: Within 1 mK above the λ point the vapor peak in the spectrum sharply reappears. This corresponds to the change in the velocity of the mode from Eq. (26) to that of Eq. (10) as the helium makes the transition from a superfluid to a classical fluid.

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