PHYSICAL REVIEW LETTERS

VOLUME 42 26 FEBRUARY 1979 NUMBER 9

Proof of the Positive-Action Conjecture in Quantum Relativity

Richard M. Schoen
Department of Mathematics, University of California, Berkeley, California 94720

and

Shing-Tung Yau
Department of Mathematics, Stanford University, Stanford, California 94305
(Received 27 November 1978)

We extend our previous method of proving the positive-mass conjecture to prove the positive-action conjecture of Hawking for asymptotically Euclidean metric. This result is crucial in proving the path integral convergent in the Euclidean quantum gravity theory.

In Hawking's[1] Euclidean approach to quantum gravity, the space of all four-dimensional Riemannian metrics is divided into conformal classes, each of which has a representative of zero scalar curvature. Then an important conjecture in this theory is the positivity of the action for any asymptotically Euclidean metric with zero scalar curvature. In this Letter, we settle this conjecture in the affirmative by extending our previous method of proving the positive-mass conjecture.[2-4] However, we emphasize here that a direct extension of our previous method will not work because of dimensional reasons. (Later on we will indicate how counter examples arise at several delicate points of our previous approach. These examples show that one cannot simply wave hands on these points without going through more delicate arguments.)

Let M be a four-dimensional asymptotically Euclidean manifold with m asymptotic regions. Then in each asymptotic region there is a coordinate chart such that the metric is given by

$$(1+4I/3\pi r^2)(dx^2+dy^2+dz^2+dt^2)+O(r^{-3}), \qquad (1)$$

where $r^2=x^2+y^2+z^2+t^2$. Our method shows that if the Ricci scalar curvature of M is nonnegative everywhere, then I is positive unless M is the flat Euclidean metric. This, of course, settles the positive-action conjecture in the affirmative.

Below we outline the procedure of the demonstration. The delicate points will be discussed here and will be presented in detail elsewhere.

As in Ref. 4, we divide the proof into two parts. The first step is to prove that the action is nonnegative. The second step is to prove that if the action is zero, then the metric is flat. Both of these two parts are much more subtle than the previous paper because of the dimension reason.

As in Ref. 4, in the first step, we assume that the action is negative and then derive a contradiction. In this case, we conformally deform our metric as in Ref. 4 to make the scalar curvature $R>0$ outside a compact set. At this point one would think that one can argue as in Ref. 4 to establish the existence of a complete minimal hypersurface lying between two hyperplanes by solving the boundary-value problem and taking the limit. This turns out to be a *serious* mistake because of dimensional reasons. In Ref. 4, the existence of such a hypersurface comes from the negativity of the mass. However, in the case under consideration here, we can prove the existence of such a hypersurface whether the action is negative or not. Hence, the existence of such an object will not create a contradiction to the negativity of the action in contrast to the argument in Ref. 4.

In order to overcome this difficulty, we solve the *free* boundary-value problem for the minimal hypersurface in an expanding sequence of cylinders in the asymptotically Euclidean region. We will present the existence proof elsewhere (which is by no means trivial). Then also by a rather *nontrivial* analysis, we show that the complete minimal hypersurface H so constructed is still asymptotically flat in the sense of Ref. 4 and has vanishing Arnowitt-Deser-Misner[5] mass. [Asymptotically flat in the sense of Ref. 4 means that H is diffeomorphic to R^3 and the metric has the form $(1+m/2r)^4\delta_{ij}+O(r^{-2})$.]

Since H is obtained by a minimizing procedure, we know that the second variation of it must be nonnegative. By a similar manipulation as in Ref. 4 and using the fact that the scalar curvature of M is positive outside a compact set, we know that

$$\int_H (|\nabla f|^2+\bar{R}f^2)\geq 0, \tag{2}$$

where $\bar{R}$ is the scalar curvature of H and f belongs to the class of smooth functions which are asymptotically constant at infinity of H. Furthermore, when f is nonzero outside a compact set of H, we can assume (2) to be a strict inequality. (The advantage of our way of producing H is that we have a wider class of f whereas in the former approach, f must approach zero at infinity of H.)

Clearly, (2) implies

$$\int_H (8|\nabla f|^2+\bar{R}f^2)\geq 0. \tag{3}$$

Now, we multiply the metric of H by u^4, where u is a function solving the equation (with boundary values asymptotic to 1 at infinity)

$$-\nabla^2 u+\tfrac{1}{8}\bar{R}u=0. \tag{4}$$

The resulting metric has zero scalar curvature, and by studying Eq. (4) *carefully*, we can prove that the Arnowitt-Deser-Misner mass of H is given by

$$-(4\pi)^{-1}\int_H (|\nabla u|^2+\tfrac{1}{8}Ru^2), \tag{5}$$

which is negative by (3) and the fact that u is not zero outside a compact set of H. This contradicts our theorem on the positivity of the mass.

In step 2, we can argue, as in Ref. 4, that when the action is zero the Ricci tensor of M must be identically zero. In three dimensions, this implies that M is flat. In our case, the argument is not trivial. If H is compact, one may try to argue by some variation formula of H. However, when H is not compact, this is not possible and we proceed as follows. Using the assumption that M is asymptotically Euclidean, we construct a geodesic line in M so that each segment of this geodesic minimizes length. Then one can use the Ricci flat assumption to prove that M is flat.

Some of our arguments have worked for locally asymptotically flat manifolds and we shall discuss this case elsewhere.

This research was supported in part by the National Science Foundation under Grant No. MCS78-04872.

[1]S. W. Hawking, Phys. Rev. D 18, 1747 (1978).

[2]R. Schoen and S.-T. Yau, Proc. Nat. Acad. Sci. U. S. A. 75, 2567 (1978).

[3]R. Schoen and S.-T. Yau, "Existence of Incompressible Minimal Surfaces and the Topology of Three Dimensional Manifolds with Non-negative Scalar Curvature" (to be published).

[4]R. Schoen and S.-T. Yau, "On the Proof of the Positive Energy Conjecture in General Relativity" (to be published).

[5]R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 118, 1100 (1960).