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Generation of Asymptotically Flat, Stationary Space-Times with Any Number of Parameters

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A method is presented for generating stationary, axisymmetric, asymptotically flat, vacuum solutions of Einstein's equation with an arbitrary number of mass and angular-momentum parameters. As an example, a two-parameter asympotically flat solution, generalizing extreme Kerr, is generated from flat space. It is conjectured that the method, applied to the general static metric, can be used to generate all stationary, axisymmetric, asymptotically flat metrics.

In previous papers[1,2] the symmetry group of stationary axisymmetric gravitational fields was explored. In a given space-time one defines a hierarchy of potentials $N_{AB}{}^{(m,n)}$ on which the transformations $\gamma_{AB}{}^{(k)}$ of the symmetry group K act ($A,B=1,2;\ m,n,k=0,1,2,\ldots$). The potential uniquely determines the space-time, being related to the Ernst potential $\mathcal{E}$ via $\mathcal{E}=iN_{11}{}^{(0,1)}$.

In this Letter we present a method which, utilizing the existence of these symmetries, can be used to generate stationary, axisymmetric, asymptotically flat space-times with any number of parameters. It involves lengthy but straightforward calculations. Applied even to flat space-time it gives interesting solutions. Compared to the method suggested in a previous Letter[3] it is much more powerful, mainly because in the present method the exponentiation of the infinitesimal transformations can be performed easily.

In the previous method one concentrates on the commuting transformations $\gamma_{22}{}^{(k+2)}+\gamma_{11}{}^{(k)}$, which leave Minkowski space invariant. We here concentrate on certain infinite linear combinations of the $\gamma_{22}{}^{(k)}$, again commuting, which map flat space into an asymptotically flat space. The action of these infinitesimal transformations on $N_{11}{}^{(m,n)}$ is given by (from now on we drop the subscripts of $\gamma_{22}{}^{(k)}$ and $N_{11}{}^{(m,n)}$,

$$N^{(m,n)}\to N^{(m,n)}+\sum_{k=0}^{\infty}\gamma^{(k)}\sum_{j=1}^{k}N^{(m,j)}N^{(k-j,n)}. \tag{1}$$

Introducing the matrices $N=\{N^{(m,n)}\}$, $A=\{A^{(i,j)}=\gamma^{(i+j)}\}$, the transformation (1) can be written as $N\to N+NAN$. This can be exponentiated yielding, for finite A, $\tilde{N}=(I-NA)^{-1}N$ or, in terms of the matrix elements,

$$\tilde{N}^{(m,n)}-N^{(m,n)}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{N}^{(m,i)}\gamma^{(i+j)}N^{(j,n)}. \tag{2}$$

Choosing

$$\gamma^{(k)}=\alpha_p\frac{k!}{(k-p)!}u^k \tag{3}$$

for every nonnegative integer rank p and constants α_p and u and using the generating function

$$G(s,t)=\sum_{m,n=0}^{\infty}N^{(m,n)}s^mt^n$$

for the potentials, Eq. (2) leads to a linear system of equations for ${}_a\tilde{G}_b(s,t)$, where

$${}_aG_b(s,t)=s^at^b\partial_s{}^a\partial_t{}^bG(s,t).$$

Solving this system one can easily obtain the generating function of the new solution. Since $\mathcal{E}$

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$=i\partial_t G(s,t)|_{(s,t)=(0,0)}$, the new Ernst potential will be

$$\tilde{\mathcal{E}}=\mathcal{E}+i\alpha_p\Delta^{-1}\sum_{l=0}^{p}\binom{p}{l}A_l(u)\partial_{t\,p-l}G_0(u,t)|_{t=0}, \qquad (4)$$

where Δ is the $(p+1)\times(p+1)$ determinant

$$\Delta=\det\left\{\delta_{il}-\alpha_p\binom{p}{p-l}_{p-l}G_i(u,u)\right\} \qquad (5)$$

and $A_l(u)$ is the determinant obtained by replacing the lth column in (5) by the column $\{_0G_i(0,u)\}$.

It should be pointed out that the transformations $\gamma_{22}^{(k)}$ commute and hence the present transformations also commute. Successive applications of transformations of different rank p with different constants α_p and u lead to additional new solutions. The most interesting seems to be the application of the combined transformation with constants

$$\gamma^{(k)}=\sum_{\lambda=0}^{p}\alpha_\lambda\frac{k!}{(k-\lambda)!}u^k \qquad (6)$$

which introduces $p+2$ new parameters. Expressions analogous to Eqs. (4) and (5) can be written easily.

The present transformations preserve asymptotic flatness in the sense that the Riemann tensor of the new solutions approaches zero for large r. An angular-momentum monopole, like the one in the Newman-Unti-Tamburino (NUT) metric, does arise but it can be eliminated via an additional Ehlers transformation.

As an example, we apply the transformations to flat space. The generating function in this case is

$$G(s,t)=\frac{-it}{2S(t)}\left(1+\frac{s+t-4stz}{sS(t)+tS(s)}\right), \qquad (7)$$

where $S(t)=[(1-2t\rho)^2+4t^2z^2]^{1/2}$, and ρ and z are the usual cylindrical coordinates. The combined action of a rank-zero and a rank-one transformation yields the solution with

$$\xi=\frac{1-\mathcal{E}}{1+\mathcal{E}}=\frac{\alpha_0r^3-2\alpha_1r^2\cos\theta-2i\alpha_1^2\sin^2\theta\cos\theta}{r^4-i\alpha_0r^3\cos\theta+2i\alpha_1r^2(\cos^2\theta-\sin^2\theta)-\alpha_1^2\sin^2\theta(1+\cos^2\theta)}, \qquad (8)$$

where r and θ are spherical coordinates. (It turns out that the coordinate freedom $z\to z+\text{const}$ can be used to eliminate the parameter u.) For $\alpha_1=0$ the above expression reduces to the expression for the extreme Kerr metric. It should be mentioned that the pth rank transformation with $p\geq 1$ applied to flat space leads to massless dipole solutions.

The general stationary, axisymmetric, asymptotically flat metric is characterized by a double infinity of parameters, which represent the (arbitrary) mass and angular-momentum multipole moments[4] of the space-time. The well-known class of static, axisymmetric Weyl metrics has arbitrary mass moments but no angular-momentum multipole moments. The present transformations applied to any Weyl metric introduces additional angular-momentum multipole moment parameters. We believe that these transformations are sufficient to generate all the stationary, axisymmetric, asymptotically flat, vacuum metrics. This is a somewhat stronger version of Geroch's conjecture.[5]

Details of the derivation of Eq. (4) and further examples will be published elsewhere.[6] This work was partially supported by National Science Foundation Grants No. PHY76-12246 and No. PHY78-12294.

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