PHYSICAL REVIEW LETTERS

VOLUME 42 | 12 FEBRUARY 1979 | NUMBER 7

Determination of the Hamiltonian in the Presence of Constraints

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(Received 2 November 1977)

The ambiguity in the Hamiltonian for systems with constraints is discussed, and a counter example is provided to a conjecture of Dirac, which has led previously to the identification of the system Hamiltonian as the extended Hamiltonian, H_E. The conjecture fails and H_E also generates the wrong canonical equations; Dirac's test for identifying the Hamiltonian, however, leads to correct results for the example.

In a well-known work,[1] Dirac has extended the applicability of Hamiltonian methods to systems for which the rank of the $N\times N$ matrix,

$$W_{nm}=\|\partial^2L(q,\mathring{q},t)/\partial\mathring{q}_n\,\partial\mathring{q}_m\|, \tag{1}$$

where L is the Lagrangian for the system, is less than N. In this case the equations of the generalized momenta,

$$p_n=\partial L/\partial\dot{q}_n,\quad n=1\text{ to }N, \tag{2}$$

produce $M>0$ functionally independent relations,

$$\varphi_m(q,p)\approx 0,\quad m=1\text{ to }M, \tag{3}$$

where Dirac's symbol for weak equality has been used. The Hamiltonian equations are found by the standard methods of the calculus of variations, with the conditions, Eqs. (3), taken into account by the method of Lagrange multipliers.[2] This produces the total Hamiltonian,

$$H_T\simeq H+\textstyle\sum_m\varphi_m u_m, \tag{4}$$

H being the usual Hamiltonian and the u_m the undetermined multipliers to Eqs. (3). In Eq. (4) "$\simeq$" has been used for strong equality. Equations (3) are primary constraints, so called because they derive directly from Eqs. (2); they are distinguished from the secondary constraints,

$$\varphi_k(q,p)\approx 0,\quad k=1\text{ to }K, \tag{5}$$

which derive from an algorithm of consistency requirements that all the constraints of the formalism be time independent, viz.

$$\dot{\varphi}_j\approx\{\varphi_j,H_T\}\approx 0,\quad j=1\text{ to }M+K\equiv J. \tag{6}$$

The J constraints fall into two classes, first class and second class, every first-class constraint having vanishing Poisson bracket with each of the remaining $J-1$ constraints. Following Dirac,[2] one can derive a general form for H_T, arriving at

$$H_T\simeq H'+\textstyle\sum_p\varphi_p v_p, \tag{7}$$

where H' is a certain first-class function of the q's and p's (i.e., it has vanishing Poisson bracket with every first-class constraint), the v_p are undetermined arbitrary functions of the time, and $\{\varphi_p\}$ is the complete set of first-class primary constraints.

The presence of the v_p in Eq. (7) introduces a novel feature: The infinitesimal changes $(\delta q,\delta p)$ of the canonical variables, from time t to $t+\delta t$, are v_p dependent under the action of H_T. Hence, the correspondence of a set of values of the (q,p) to a physical state cannot be one to one, if the characterization of a physical state be required to be free from undetermined arbitrary functions. Proceeding from considerations of this kind and based upon the form of v_p dependence that actually results, one can identify a set of quantities, which as generators of infinitesimal canonical transformations do not lead to a change of physical state. These are the first-class constraints, φ_g, needed on the right-hand sides of the equations

$$\{\varphi_g, \varphi_{g'}\} \simeq \sum_{g''} \gamma_{gg'g''}(q,p)\varphi_{g''}, \tag{8a}$$

$$\{\varphi_g, H'\} \simeq \sum_{g''} h_{gg''}(q,p)\varphi_{g''}, \tag{8b}$$

where the indices g, g', and g'' extend over a minimal range such that the complete primary first-class subset is included. Dirac has proposed, accordingly, that H_T should be replaced by a "generalized" Hamiltonian, H_g, to generate the canonical formalism. To obtain H_G the summation appearing in Eq. (7) must be extended to include all those generators which do not change the state, with all the coefficients arbitrary. Assuming that the φ_g exhaust that collection, this requirement leads to

$$H_T \rightarrow H_G \simeq H' + \sum_g \varphi_g v_g. \tag{9}$$

I will refer to Eqs. (8) as *Dirac's test.*

Dirac has conjectured that all the first-class secondary constraints belong to the set of generators which do not change the state.[2] With the assumption of completeness, this leads to a determination of the Hamiltonian as

$$H_T \rightarrow H_E \simeq H' + \sum_e \varphi_e v_e, \tag{10}$$

where now the summation extends over all the first-class primary and secondary constraints, and all the v_e are arbitrary. It is customary to refer to H_E as the extended Hamiltonian.

I will give an example with the following properties: (i) If the form on the right-hand side in Eq. (10) is used, not all the v_e are necessarily arbitrary. (ii) Dirac's test leads to an H_G different from H_E. (iii) Together with the constraints, the canonical equations from the H_G determined from Dirac's test imply precisely the original Euler-Lagrange equations; those from H_E do not. (iv) The secondary constraint terms in H_G are fixed uniquely.

Now for the example. The Lagrangian

$$L = L(x,y,z,\dot{x},\dot{y},\dot{z}) = \sum_n(\dot{x}_n\dot{z}_n + \tfrac{1}{2}y_n z_n^2) \tag{11}$$

produces the Euler-Lagrange equations

$$\ddot{z}_n = 0, \quad 0 = \tfrac{1}{2}z_n^2, \quad \ddot{x}_n = y_n z_n, \quad n = 1 \text{ to } N, \tag{12}$$

whose solutions are those of uniform x motion, confined to the x-y plane ($z=0$), and with $y(t)$ remaining undertermined, for an N-particle system. So the system corresponding to L is consistent.

There are N primary constraints,

$$p_{y_n} = \partial L/\partial \dot{y}_n \approx 0, \quad n = 1 \text{ to } N, \tag{13}$$

and the total Hamiltonian has N arbitrary functions, v_n,

$$H_T \simeq \sum_n (p_{z_n} p_{x_n} - \tfrac{1}{2} y_n z_n^2) + \sum_n p_{y_n} v_n. \tag{14}$$

The consistency conditions to Eqs. (13) are

$$\dot{p}_{y_n} \approx \{p_{y_n}, H_T\} \approx \tfrac{1}{2} z_n^2 \approx 0, \quad n = 1 \text{ to } N, \tag{15}$$

which give additional secondary constraints,

$$z_n \approx 0, \quad n = 1 \text{ to } N. \tag{16}$$

The algorithm terminates after the next step, which gives N more secondary constraints,

$$\dot{z}_n \approx \{z_n, H_T\} \approx p_{x_n} \approx 0, \quad n = 1 \text{ to } N. \tag{17}$$

All $3N$ constraints are first class. Noting that

$$z_n^2 \simeq 0, \quad n = 1 \text{ to } N, \tag{18}$$

as an immediate consequence of (16), we find that

$$H_T \simeq H' + \sum p_{y_n} v_n, \tag{19a}$$

where

$$H' \simeq \sum_n p_{z_n} p_{x_n}. \tag{19b}$$

Since the expressions (8a) and (8b) are

$$\{p_{y_n}, p_{y_{n'}}\} \simeq 0, \tag{20a}$$

$$\{p_{y_n}, H'\} \simeq 0, \tag{20b}$$

for all n, n', *none* of the first-class secondary constraints appears on the right-hand sides, and the generalized Hamiltonian from Dirac's test is the same as H_T (in this example), viz.

$$H_G \simeq H' + \sum_n p_{y_n} v_n. \tag{21}$$

Equations (16) and (17) give all the secondary first-class constraints; so the extended Hamiltonian is

$$H_E \simeq \sum_n p_{x_n} u_n + \sum_n p_{y_n} v_n + \sum_n z_n w_n, \tag{22}$$

where u_n and w_n are $2N$ additional arbitrary functions.

The canonical equations from H_G are given by

$$\mathring{x}_n \approx p_{z_n}, \quad \mathring{y}_n \approx v_n, \quad \mathring{z}_n \approx 0, \qquad \mathring{p}_{x_n} \approx 0, \quad \mathring{p}_{y_n} \approx 0, \quad \mathring{p}_{z_n} \approx 0, \tag{23}$$

which, together with the $3N$ constraints, correctly express the full content of Eqs. (12). The canonical equations from H_E, however, are

$$\mathring{x}_n \approx u_n, \quad \mathring{y}_n \approx v_n, \quad \mathring{z}_n \approx 0, \qquad \mathring{p}_{x_n} \approx 0, \quad \mathring{p}_{y_n} \approx 0, \quad \mathring{p}_{z_n} \approx -w_n, \tag{24}$$

which imply arbitrary time dependence of all the unconstrained variables. Equations (24), together with the constraints, do not imply Eqs. (12). Finally, we note that the first-class secondary constraints, z_n and p_{x_n}, are infinitesimal generators for translations of p_{z_n} and x_n, whose time dependence *is* determined from the dynamics, and which does correspond to a change of physical state.

Clearly the $3N$ arbitrary functions in Eq. (22) are not needed. Worse, "gauge freedom" has turned to license since *none* of the dynamical information in Eqs. (12) survives there with the journey from the Lagrange formalism. H_E is the wrong choice for the Hamiltonian. The form in Eq. (21) has the right properties, however. If we proceed now in the spirit of Dirac's treatment of the electromagnetic field[2] and also of the gravitational field,[3] the second term in Eq. (21) can be dropped, for there was no dynamical content to the y motion to start with and the term in question "commutes" with H'. The motion then is played out on a reduced, $4N$-dimensional phase space, with the (y, p_y) sector gone.

Thus, the determination of the Hamiltonian for a system with constraints can be seen to proceed in two steps. In the first, a suitable H_G is found; in the second, more or less optional, step, unneeded variables are dropped, such as the (y, p_y) set in the example. Dirac has stressed, in regard to step one, the importance of passing from H_T to H_G so that the full gauge-transforming power of the Euler-Lagrange formalism will be realized in the canonical formalism. The example of the present paper shows that it also is necessary to be sure, in step one, that the canonical equations guarantee the original Euler-Lagrange equations, as this feature otherwise might be lost.

There is an important observation to be made concerning the definition of the constraint functions, φ_j, selected to implement the consistency conditions. Equation (16) was chosen to express the result of Eq. (15); thus the quantity to which Eq. (6) was (again) applied was taken to be z_n, resulting in Eq. (17). Some other power of z_n, e.g., z_n^2, might have been selected instead. The choice made affects the equations from which H_G and H_E, both, are to be determined. More generally, one could choose z_n^α, for any $\alpha \neq 0$. The same arbitrariness exists also for the primary constraints; one might choose $p_{y_n}^\alpha$ and apply Eq. (6) to that. With choices of this kind one has, e.g.,

$$\mathring{\varphi} \approx \{z_n^\alpha, H_T\} \approx \alpha z_n^{\alpha-1}\{z_n, H_T\}. \tag{25}$$

If $\alpha > 1$, $\mathring{\varphi} \approx 0$ does not express a condition since $z_n^{\alpha-1}$ already vanishes on the constraint hypersurface; while if $\alpha < 1$, $\mathring{\varphi}$ is undefined on the constraint hypersurface. Thus the choice of the text, which is $\alpha = 1$, is singled out as the only meaningful one, $\alpha \neq 1$ being ill defined or empty. The choice $\alpha = 1$ is also in the spirit of Dirac's original prescription for the definition of weak equality,[1] that $\varphi \approx 0$ express the fact that under small variations of the coordinates, by an amount $O(\epsilon)$, φ be $O(\epsilon)$. We can accomplish this result more formally by requiring that at the rth stage of the algorithm the functions φ_j be chosen so that,

$$\det\|\partial\varphi_{j_r}(\eta)/\partial\eta_{j_r'}\| \neq 0, \qquad j_r, j_r' = 1 \text{ to } J_r = M + K_r, \tag{26}$$

where η denotes the $2N$ q's and p's collectively, and the η_{j_r} are J_r of the $2N$ coordinates. The primary subset[4] can be taken to be that for $r = 0$, with $K_0 = 0$; while at the end of the process $J_r \to J = M + K$, as in Eq. (6). In addition, the determinant in (26) should be required not to be singular on the hypersurface defined by the φ_{j_r}, at each stage.

It will be observed, incidentally, that the choice $\alpha = 1$ also can affect the Hamiltonian through H'. Thus Eq. (16) gave Eq. (18), so that H_T in Eq. (14) was simplified to the form given in Eqs. (19).

Finally, another example with the same features as that studied above is defined by the Lagrangian,

$$L = \sum_n (\mathring{x}_n \mathring{z}_n + \tfrac{1}{2} y_n \mathring{z}_n^2). \tag{27}$$

The analysis easily proceeds along similar lines.

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[1]P. A. M. Dirac, Can. J. Math. 2, 147 (1950), and Proc. Roy. Soc. London, Ser. A 246, 326 (1958).

[2]See, for example, P. A. M. Dirac's little book, *Lec-*

tures on Quantum Mechanics* (Yeshiva Univ. Press, New York, 1964).

[3]P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 246, 333 (1958).

[4]The condition, (26), is satisfied directly for the primary constraints in the treatment given by E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974), Chap. 8.

Class of "Noncanonical" Vacuum Metrics with Two Commuting Killing Vectors

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(Received 20 December 1978)

I present a class of vacuum metrics with a pair of commuting Killing vectors which do not possess the property of orthogonal transitivity. The solution is Petrov type II with expansion-, twist-, and shear-free geodesic rays and includes the van Stockum exterior metric as a limiting case.

Recently[1,2] there has been considerable progress in generating new vacuum metrics with two commuting Killing vectors from the familiar ones by utilizing the hidden symmetry properties of the reduced field equations, discovered by Geroch.[3] However, an element of this group acting on a vacuum metric yields a new one preserving the two Killing vectors $\xi_a{}^\mu$ $(a,b,c=0,1)$ if (and in general only if) the orbits generated by them are orthogonally transitive. The two scalars

$$c_a = e^{\mu\nu\rho\sigma}\xi_{0\mu}\xi_{1\nu}\xi_{a\rho;\sigma} \tag{1}$$

then vanish everywhere and the metric can be expressed in a canonical 2×2 block diagonal form. The most physically interesting class of solutions with a two-parameter Abelian group of motion are the stationary axisymmetric gravitational fields. It is well known[4] that the stationary axisymmetric space-times satisfying the vacuum field equations $R_{\mu\nu}=0$ [or more generally the relation $R_{\mu\nu}\xi_a{}^\nu = R_a{}^b(x)\xi_{b\mu}$] in a region including a portion of the symmetry axis are orthogonally transitive.

In this Letter I present a class of vacuum solutions with two commuting Killing vectors $\xi_a{}^\mu$ whose orbits *do not* admit orthogonal two-surfaces and therefore lie outside the scope of the Geroch-Kinnersley program. This indicates that the condition $c_a=0$, while not very restrictive, does exclude[5] some (possibly interesting) vacuum solutions which admit a pair of commuting Killing vectors. We define the quantities f_{ab} and f^{ab} by

$$f_{ab} = \xi_{a\mu}\xi_b{}^\mu, \quad f_{ab}f^{bc} = \delta_a{}^c. \tag{2}$$

The vacuum field equations assume a readily integrable form when the two constants c_0 and c_1 satisfy the additional condition

$$f^{ab}c_a c_b = 0. \tag{3}$$

By choosing a suitable linear combination of these two vectors we can set $c_1=0$, $c_0\neq 0$. When $r^2 \equiv -\det(f_{ab}) \neq \text{const}$, the solution turns out to be

$$ds^2 = (r\psi - \alpha^2 r^{-1/2})\,du^2 + 2du\,(r\,d\varphi + \alpha r^{-1/2}dz) - r^{-1/2}(dr^2 + dz^2), \tag{4}$$

where $\psi(r,z)$ is an arbitrary solution of the three-dimensional Laplace's equation in cylindrical coordinates (comma denotes ordinary differentiation):

$$r^{-1}(r\psi_{,r})_{,r} + \psi_{,zz} = 0 \tag{5}$$

and $\alpha = \frac{2}{3}c_0$. The metric is singular at $r=0$; the square of the Riemann tensor, viz., $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, diverges as r^{-3} as $r\to 0$. It is type II in the Petrov-Pirani classification and the (doubly degenerate) propagation vector, which is also a Killing vector, is in either case tangent to a congruence of expansion-, twist-, and shear-free null geodesics ($K=\rho=\sigma=0$ in the Newman-Penrose notation). When $\alpha\to 0$ it reduces to the van Stockum exterior metric which seems to be the only nondiverging ($\rho=0$) type-II vacuum solution known in the literature.[2] When $\alpha\neq 0$, it is a redundant parameter and may be set equal to unity by rescaling the Killing vectors. The solution can be derived systematically using the methods of Ref. 6. This derivation and further details will be given elsewhere.

It is a pleasure to thank Professor S. Okubo for his encouragement during the progress of this work and many helpful comments. This work