PHYSICAL REVIEW LETTERS

VOLUME 42 | 1 JANUARY 1979 | NUMBER 1

One-Particle Reduced Density Matrix of Impenetrable Bosons in One Dimension at Zero Temperature

H. G. Vaidya
Department of Physics, Columbia University, New York, New York 10027

and

C. A. Tracy
Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755
(Received 10 October 1978)

We report the large-r and small-r expansions of the one-particle reduced density matrix $\rho(r)$ of a system of impenetrable bosons in one dimension at zero temperature. These expansions were derived from an exact calculation of $\rho(r)$ to be reported elsewhere. We find that the large-r expansion of $\rho(r)$ contains oscillatory terms which we relate to the analytic structure of the momentum density function $n(k)$.

One of the model systems that has continued to attract interest is the nonrelativistic many-body system of bosons in one dimension interacting through a two-body potential $c\delta(x_i - x_j)$.[1] In particular, the limit $c \to \infty$ corresponds to a gas of impenetrable bosons. The exact ground-state wave function of N impenetrable bosons on a chain of length L with periodic boundary conditions was shown by Girardeau[2] to be given by

$$\psi_{N,L}(x_1, \ldots, x_N) = (N!L^N)^{-1/2} \prod_{1 \le n < m \le n} \left| \exp\left(\frac{2\pi i}{L} x_n\right) - \exp\left(\frac{2\pi i}{L} x_m\right) \right|. \tag{1}$$

The study of the one-particle reduced density matrix (which we henceforth refer to simply as the density matrix), defined (for zero temperature) by

$$\rho_{N,L}(x - x') = N \int_0^L dx_1 \cdots \int_0^L dx_{N-1} \psi_{N,L}(x_1, \ldots, x_{N-1}, x) \psi_{N,L}{}^*(x_1, \ldots, x_{N-1}, x'), \tag{2}$$

was initiated by Schultz[3] and by Lenard.[4] In this Letter we report the large-r and small-r expansions of $\rho(r)$ obtained from an exact calculation of this density matrix for the system of impenetrable bosons in the thermodynamic limit and at zero temperature.

The thermodynamic limit is the limit $N \to \infty$, $L \to \infty$, such that $\rho = N/L$ is fixed. Denoting the thermodynamic limit of $\rho_{N,L}(x - x')$ by $\rho(x - x')$, we know from the work of Lenard[4] that this limit exists, that the limiting momentum distribution function $F(k)$ exists and is a continuous function of k, and that these two quantities are related by

$$\rho(x) = \int_{-\infty}^{\infty} e^{ikx} dF(k). \tag{3}$$

We will write $dF(k) = n(k)dk$, where $n(k)$ is the limiting momentum density function. Both Schultz and

Lenard proved that $\lim_{x\to\infty}\rho(x)=0$. This means that there is no off-diagonal long-range order[5] which in turn implies[5] that there is no macroscopic occupation of the $k=0$ state.

For $x\to\infty$, we find[6] that $\rho(x)$ has the following asymptotic expansion[7]:

$$\rho(x)=\rho_\infty|x|^{-1/2}\left[1+\frac{1}{8}\left(\cos(2x)-\frac{1}{4}\right)\frac{1}{x^2}+\frac{1}{8}\left(\frac{13}{8}+\frac{1}{\pi}\right)\frac{\sin(2x)}{x^3}+\frac{1}{32}\left(\frac{1}{\pi}-\frac{159}{256}\right)\frac{1}{x^4}-\frac{1}{64}\left(\frac{7}{\pi}+\frac{181}{8}\right)\frac{\cos(2x)}{x^4}+O(x^{-5})\right], \tag{4}$$

with $\rho_\infty=\pi e^{1/2}2^{-1/3}A^{-6}=0.92418\ldots$ and $A=1.2824271\ldots$ is Glaisher's constant. Two aspects of (4) should be noted: (i) the algebraic decay of $\rho(x)$ and (ii) the presence of oscillatory terms in the correction terms to the leading $x^{-1/2}$ behavior.

Lenard[4] has derived an expansion of $\rho(x)$ for small x. We have used these results to extend the expansion to order x^9 [Lenard expanded $\rho(x)$ to order x^4] with the result[7]

$$\rho(x)=1-\frac{1}{6}x^2+\frac{1}{9\pi}|x|^3+\frac{1}{120}x^4-\frac{11}{1350\pi}|x|^5-\frac{1}{5040}x^6+\frac{122}{105\pi\cdot 7!}|x|^7+\left(\frac{1}{18300\pi^2}-\frac{1}{55\cdot 8!}\right)x^8-\frac{2741}{1575\cdot 99\pi\cdot 7!}|x|^9+O(x^{10}). \tag{5}$$

Using the expansions (4) and (5) (these expansions overlap to within 1% for $x\approx 2.7$), we plot $\rho(x)$ in Fig. 1 along with Lenard's upper bound $(e/x)^{1/2}$.

The expansion of $\rho(x)$ for large x has the general asymptotic structure

$$\rho(x)=\frac{\rho_\infty}{|x|^{1/2}}\left[1+\sum_{n=1}^{\infty}\frac{c_{2n}}{x^{2n}}+\sum_{m=1}^{\infty}\frac{\cos(2mx)}{x^{2m}}\left(\sum_{n=0}^{\infty}\frac{c_{2n,m}'}{x^{2n}}\right)+\sum_{m=1}^{\infty}\frac{\sin(2mx)}{x^{2m+1}}\left(\sum_{n=0}^{\infty}\frac{c_{2n,m}''}{x^{2n}}\right)\right], \tag{6}$$

where c_{2n}, $c_{2n,m}'$, and $c_{2n,m}''$ are constants. This expansion enables us to study the singularity structure of the one-particle momentum density function $n(k)$. From (3), (6), and standard asymptotic methods we conclude that the $|x|^{-1/2}$ behavior of $\rho(x)$ at infinity leads to a $|k|^{-1/2}$ singularity in $n(k)$ at $k=0$. Furthermore, the terms with $\cos(2mx)$ and $\sin(2mx)$ in (6) lead to additional points of nonanalyticity for $n(k)$ at $k=\pm 2mk_F$, $m=1,2,\ldots$. At these points some higher derivative of $n(k)$ diverges. For example, at $k=\pm 2k_F$, $d^2n(k)/dk^2$ is divergent. All the singularities are square-root branch points.

We conclude with the following remarks.

(1) Note that a system of free fermions has a sharp Fermi surface at zero temperature whereas the system of impenetrable bosons has only a divergence in the second derivative of $n(k)$ at $k=\pm 2k_F$ and milder divergences at $\pm 2mk_F$, $m=2,3,\ldots$.

(2) The model of Sutherland[8] is a generalization of the impenetrable-boson system. For the Sutherland model the ground-state wave function is of the form of Eq. (1) where now the absolute value is raised to the power $\lambda=\lambda(g)$, g being a coupling constant appearing in the Sutherland Hamiltonian. For $\lambda=2$ bosons, $n(k)$ has, as was shown by Sutherland,[8] the simple form

$$n(k)=\begin{cases}(4\pi)^{-1}\ln(2k_F/|k|), & |k|\le 2k_F,\\ 0, & |k|\ge 2k_F.\end{cases} \tag{7}$$

Note that $\lambda=2$ bosons exhibit in $n(k)$ a logarithmic singularity at $k=0$, $n(k)$ has additional points of nonanalyticity at $k=\pm 2k_F$, and $n(k)$ is continuous at $\pm 2k_F$. For the general-λ bosons (except possibly for $\lambda=2n$, $n=1,2,\ldots$)[9] we expect that the corresponding $n(k)$ will have singularities at $k=\pm 2mk_F$, $m=0,1,2,\ldots$. We conjecture that the nature of these singularities will be a function of the parameter λ.

(3) The reduced density matrix for nonzero temperature has been studied by Lenard[10] and by Efetov and Larkin.[11] Lenard has shown how to generalize the short-distance expansion to the case $T>0$. Furthermore, Lenard showed that the $T\to 0$ limit and the thermodynamic limit commute. Lenard conjectured that for $T>0$, $\rho(r)$ should decrease exponentially as $r\to\infty$. Efetov and Larkin showed that at low temperatures this is the case with a correlation length inversely proportional to the temperature. Thus from these results we may conclude that for $T>0$ the $k^{-1/2}$ singularity at the origin in $n(k)$ will disap-

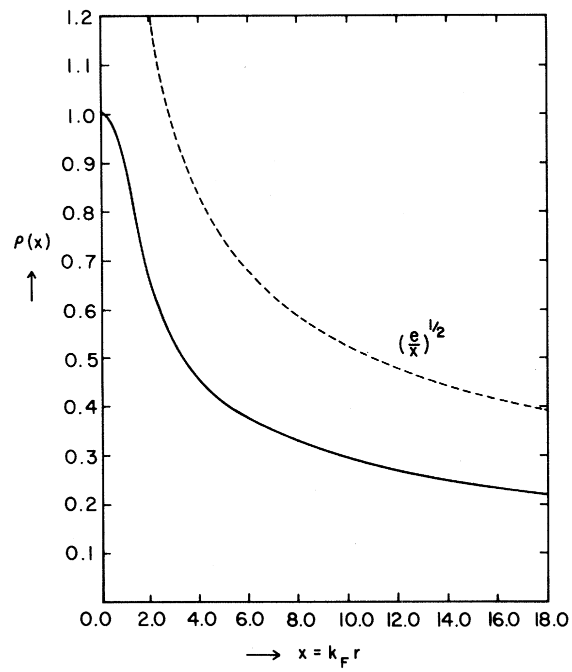

FIG. 1. $\rho(x)$ as a function of x. The dotted line is a plot of Lenard's upper bound.

pear and the branch points located for zero temperature at $\pm 2mk_F$ will move off of the real axis for $T > 0$. There is an obvious scaling function associated with these two regimes. This scaling function deserves further study, and the $\lambda = 2$ bosons of Sutherland[8] is no doubt a simpler case to examine.[11]

(4) Luther and Peschel[12] have examined the zero-temperature correlation functions of the one-dimensional spin-$\frac{1}{2}$ Heisenberg-Ising model. From their analysis they concluded that the XX correlation function behaves as $R^{-\eta}$ for large separation, where the exponent η depends upon the coupling J_z in the Heisenberg-Ising Hamiltonian (see also Efetov and Larkin[11]). These results together with comment (3) suggest that for the δ-function model,[1] $\rho(r)$ will have a power-law decay for $0 < c \leq \infty$ ($c = 0$ is free bosons) with an exponent depending upon c. Furthermore, we expect that the branch points of $n(k)$ will remain at $\pm 2mk_F$, $m = 1, 2, \ldots$, but the nature of these singularities will depend upon the value of c.

(5) Alternatively, one may view the δ-function model as the one-space, one-time quantum field theory of a complex scalar boson field $\Phi(x)$ with Lagrangian

$$\mathcal{L} = \tfrac{1}{2} i \Phi^* \overleftrightarrow{\partial}_0 \Phi - (\partial_1 \Phi^*)(\partial_1 \Phi) - c\Phi^*\Phi^*\Phi\Phi.$$

This viewpoint has been particularly emphasized by Thacker.[1] In this language the (time-dependent) density matrix $\rho(x - x', \tau)$ is the single-particle propagator $G(x - x', \tau)$. Note that $\mathcal{L}$ is U(1) invariant. The relativistic U(1) model is obtained by replacing the nonrelativistic kinetic energy operator in the above $\mathcal{L}$ by a relativistic kinetic energy operator. The strong-coupling limit of this relativistic U(1) model (first an analytic continuation to Euclidean space is performed) is the much studied[13] classical XY model in two dimensions scaled to its Kosterlitz-Thouless transition temperature T_c. We do not believe that the calculations of Ref. 13 are refined enough to ascertain the presence or absence of oscillatory terms in the correction terms to the leading $R^{-\eta}$ behavior ($T < T_c$). We conjecture, based upon our above analysis, that such oscillatory terms are present and their presence, as above, drastically affects the analytic properties of the Fourier transform of the spin-spin correlation function of the classical XY model in two dimensions ($T < T_c$).

The authors wish to acknowledge the many invaluable discussions with Professor Barry M. McCoy. Furthermore, we wish to thank Professor Chen Ning Yang for his support at the Institute for Theoretical Physics at Stony Brook, where this work was carried out, and Professor Bill Sutherland for an explanation of the work of Ref. 8. This work was supported in part by National Science Foundation Grants No. PHY-76-15328 and No. DMR 77-07863-A01.

[1]E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963); E. H. Lieb, Phys. Rev. 130, 1616 (1963); J. B. McGuire, J. Math. Phys. 5, 622 (1964); C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967); C. N. Yang and C. P. Yang, J. Math. Phys. 10, 1115 (1969); M. Gaudin, J. Math. Phys. 12, 1674, 1677 (1971); H. B. Thacker, Phys. Rev. D 11, 838 (1975), and 14, 3508 (1976), and 17, 1031 (1978).

[2]M. Girardeau, J. Math. Phys. 1, 516 (1960).

[3]T. D. Schultz, J. Math. Phys. 4, 666 (1963).

[4]A. Lenard, J. Math. Phys. 5, 930 (1964).

[5]O. Penrose and L. Onsager, Phys. Rev. 104, 576 (1956); C. N. Yang, Rev. Mod. Phys. 34, 694 (1962).

[6]H. G. Vaidya and C. A. Tracy, to be published.

[7]In Eq. (4) one should think of x as the dimensionless quantity $k_F r$, where k_F is the Fermi momentum (Ref. 2). In our calculation we have set $k_F = 1$ whereas Lenard chose $k_F = \pi$. This makes Eq. (5) appear somewhat different from the corresponding equation in Ref. 4.

[8]B. Sutherland, J. Math. Phys. 12, 246, 251 (1971), and Phys. Rev. A 4, 2019 (1971), and 5, 1372 (1972).

Additional systems with a ground-state wave function of the product type of Eq. (1) can be found in B. Sutherland, Phys. Rev. Lett. 34, 1083 (1975), and 35, 185 (1975).

[9]We thank B. Sutherland for alerting us to this point.

[10]A. Lenard, J. Math. Phys. 7, 1268 (1966).

[11]K. B. Efetov and A. I. Larkin, Zh. Eksp. Teor. Fiz. 69, 764 (1975) [Sov. Phys. JETP 42, 390 (1976)].

[12]A. Luther and I. Peschel, Phys. Rev. B 12, 3908 (1975).

[13]V. L. Berezinskii, Zh. Eksp. Teor. Fiz. 59, 907 (1970), and 61, 1144 (1971) [Sov. Phys. JETP 32, 493 (1971), and 34, 610 (1972)]; J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973); J. M. Kosterlitz, J. Phys. C 7, 1046 (1974); J. Zittarz, Z. Phys. 23B, 55, 63 (1976); A. Luther and D. J. Scalapino, Phys. Rev. B 16, 1153 (1977); J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B 16, 1217 (1977).

Measurement of the Branching Ratios for $\tau \to \pi\nu_\tau$ and $\tau \to \mu\nu_\mu\nu_\tau$

W. Bacino, T. Ferguson, L. Nodulman, W. Slater and H. Ticho
Physics Department, University of California, Los Angeles, California 90024

and

A. Diamant-Berger,[a] G. Donaldson, M. Duro, A. Hall,[b] G. Irwin, J. Kirkby
F. Merritt, and S. Wojcicki
Stanford Linear Accelerator Center and Physics Department, Stanford University, Stanford, California 94305

and

R. Burns,[c] P. Condon,[d] and P. Cowell[e]
Physics Department, University of California, Irvine, California 92664

and

J. Kirz
Physics Department, State University of New York, Stony Brook, New York 11794
(Received 26 October 1978)

On the basis of a sample of 41 events of the type $e^+e^- \to e^\pm X^\mp$ $(X \neq e)$ and no observed photons, we have observed a clear signal of the decay $\tau^- \to \pi^-\nu_\tau$. We measure the branching ratio for this decay to be $b_\pi = 0.080 \pm 0.032 \pm 0.013$ and for the decay $\tau^- \to \mu^-\bar{\nu}_\mu\nu_\tau$, $b_\mu = 0.21 \pm 0.05 \pm 0.03$, where the first and second errors are, respectively, statistical and systematic. Both measurements agree with theoretical values derived under the assumption that the τ decays via the standard weak current.

The original conjecture of a third charged lepton, proposed by Perl *et al.*[1] after their observation of anomalous $e\mu$ events, has been reinforced by subsequent detailed studies conducted at DORIS and SPEAR. With one exception all the information provided by branching-ratio measurements, lepton spectra, and the energy dependence of the production cross section have confirmed the hypothesis that the τ is a sequential heavy lepton which decays via the standard weak current.[2]

The exception[3] was a measurement of the branching ratio b_π for the decay[4] $\tau^- \to \pi^-\nu_\tau$ substantially below the theoretical expectation. From the relative rates for $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$ the standard model predicts $b_\pi/b_e = 0.59$ $(b_\pi = 0.094$ for $b_e = 0.16$, where b_e is the branching ratio for $\nu^- \to e^-\bar{\nu}_e\nu_\tau)$. The experimental measurement was reported in two forms: firstly, $b_e b_\pi = 0.004 \pm 0.005$ $(b_\pi = 0.025 \pm 0.031$ for $b_e = 0.16)$ or, alternatively, an observation of two $e\pi$ events when 7.3 were expected, based on the detection of twelve $e\mu$ events. (The latter form is insensitive to an error in b_e.)

Accordingly we have made a measurement of b_π[5] from data obtained at SPEAR using the DELCO detector. The data were taken with the apparatus described previously[6] after the addition of two muon walls (Fig. 1). The Pb walls, followed by magnetostrictive wire spark chambers (WSC) and scintillation counters, provide muon identification over 20% of 4π sr. A particle must traverse typically 2 absorption lengths of material to be tagged as a muon. This represents the best compromise between hadron discrimination and muon range at these low energies. A track is identified as a muon if it aims within a restricted sensitive